Mathematical Approach to Climate Change Impacts - MAC2I

INdAM, La Sapienza, March 2017

The Mathematics of Climate Change and of its Impacts

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Based on joint work with many collaborators, in math, climate sciences & macroeconomics (*)

ENS

(*) *Please visit these sites for more info.*

https://dept.atmos.ucla.edu/tcd

http://www.environnement.ens.fr/

Dynamical systems and predictability

- The initial-value problem → *numerical weather prediction* (*NWP*)

  — **easiest**!
- The asymptotic problem → long-term climate

  — **a little harder**
- The intermediate problem → low-frequency variability (LFV) –

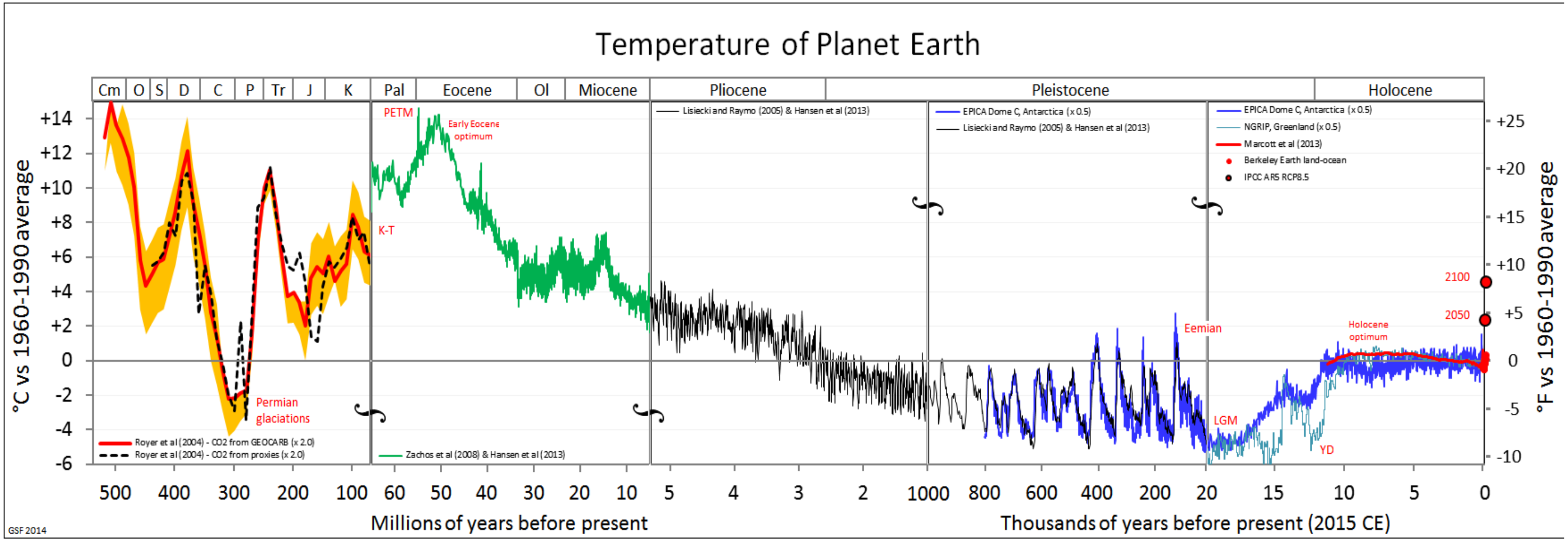
  – multiple equilibria, long-periodic oscillations, intermittency, slow transients, “tipping points”

  — **hardest!!**

Paraphrasing John von Neumann, in
R. L. Pfeffer (ed.), *Dynamics of Climate* (Pergamon, 1960)
now re-edited as an Elsevier E-book

Long-term temperature evolution on Earth

Not only do **global temperatures** move up & down on geological time scales, nor do they just switch from one long-term mean to another: They clearly show **changes in dynamic regime** — from high to low variability, from one dominant periodicity to another, from high to low drift, and so on.



Overall, to model this **complex behavior** we do need to consider both **chaotic** & **random** ingredients, both **intrinsic** & **forced** variability.

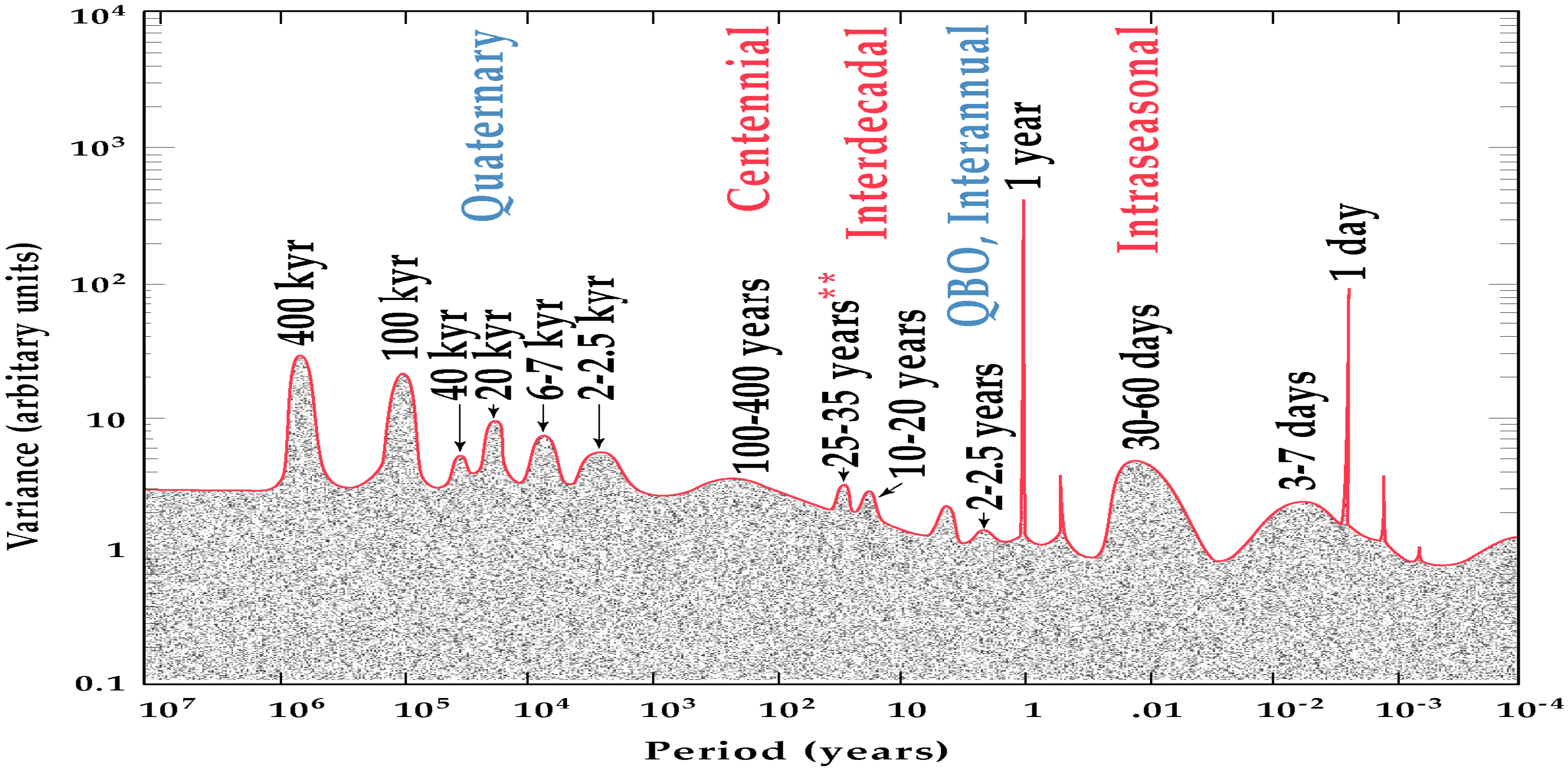
Compiled by Glen Fergus, https://commons.wikimedia.org/wiki/File%3AAll_palaeotemps.png

N.B. Plot is ~"log-linear": time axis is logarithmic+linear, temperature axis is linear.

Composite spectrum of climate variability

Standard treatement of frequency bands:

1. ***High frequencies – white noise (or “colored”)***
2. ***Low frequencies – slow evolution of parameters***



From **Ghil (2001, *EGEC*)**, after **Mitchell* (1976)**

*** “No known source of deterministic internal variability”**

**** 27 years – Brier (1968, *Rev. Geophys.*)**

Climate and Its Sensitivity

Let's say CO_2 doubles:

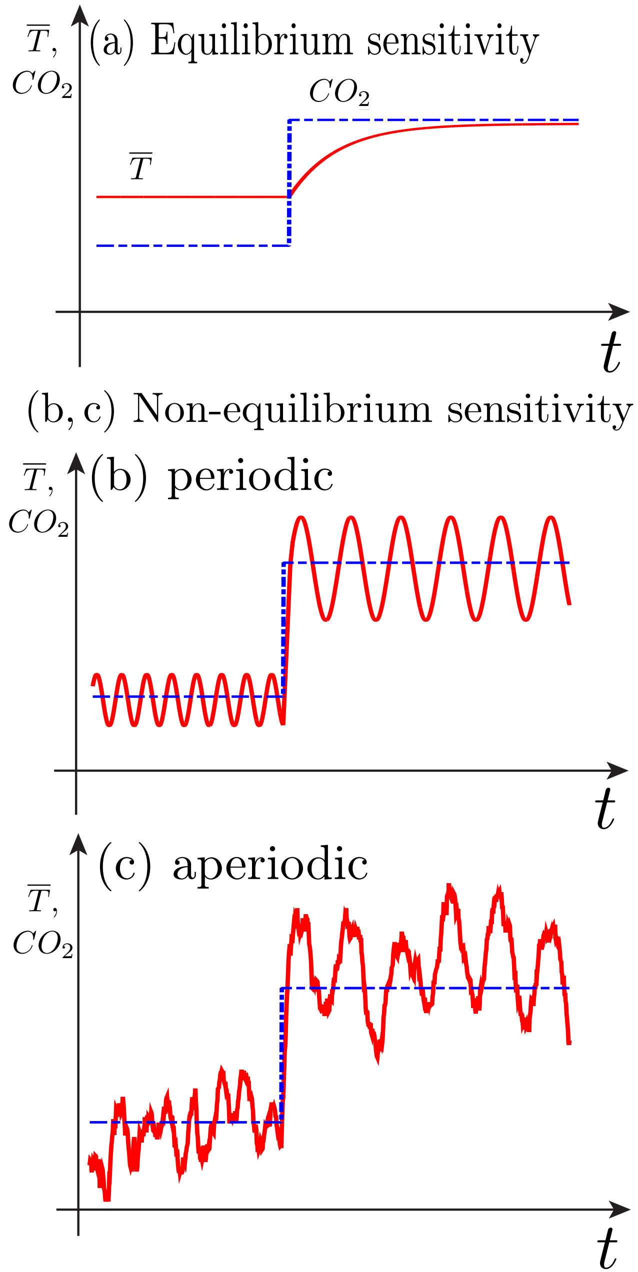
How will "climate" change?

1. Climate is in stable equilibrium (fixed point); if so, mean temperature will just shift gradually to its new equilibrium value.

2. Climate is purely periodic; if so, mean temperature will (maybe) shift gradually to its new equilibrium value. But how will the period, amplitude and phase of the limit cycle change?

3. And how about some "real stuff" now: chaotic + random?

Ghil (in *Encycl. Global Environmental Change*, 2002)



Motivation

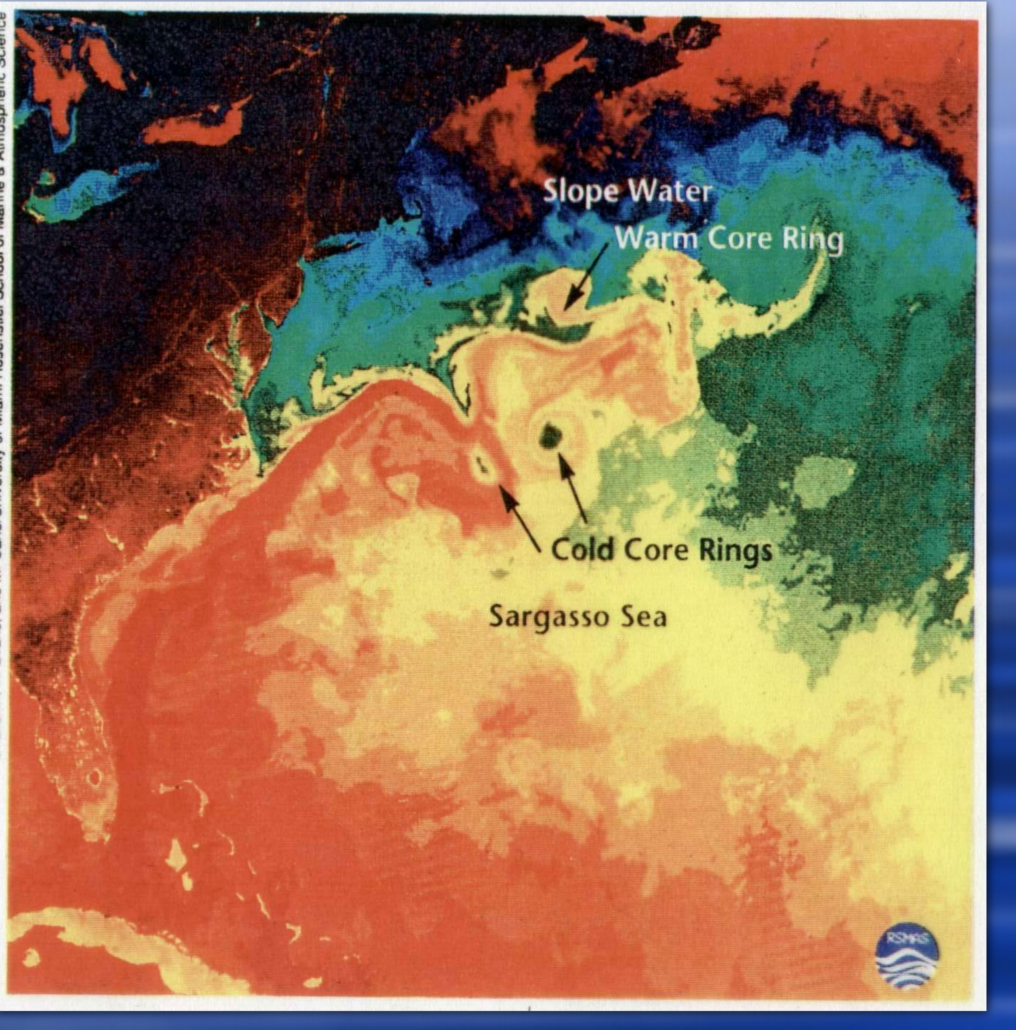
- The ***climate system*** is highly ***nonlinear and*** quite ***complex***.
- The system's ***major components*** — the atmosphere, oceans, ice sheets — ***evolve*** on many time and space scales.
- Its ***predictive understanding*** has to rely on the system's physical, chemical and biological modeling,

  but also on the thorough mathematical analysis of the models thus obtained: ***the forest vs. the trees***.
- The ***hierarchical modeling*** approach allows one to give proper weight to the understanding provided by the models *vs*. their realism: back-and-forth between ***"toy"*** (conceptual) and ***detailed*** ("realistic") ***models***, and between ***models*** and ***data***.
- How do we disentangle ***natural variability*** from ***the anthropogenic forcing***: ***can we & should we, or not?***

The gyres and the eddies

Many scales of motion, dominated in the mid-latitudes by (i) ***the double-gyre circulation***; and (ii) ***the rings and eddies***.

Much of the focus of physical oceanography over the ‘70s to ‘90s has been with the “***meso-scale***”: the meanders, rings & eddies, and the associated two-dimensional and quasi-geostrophic ***turbulence***.

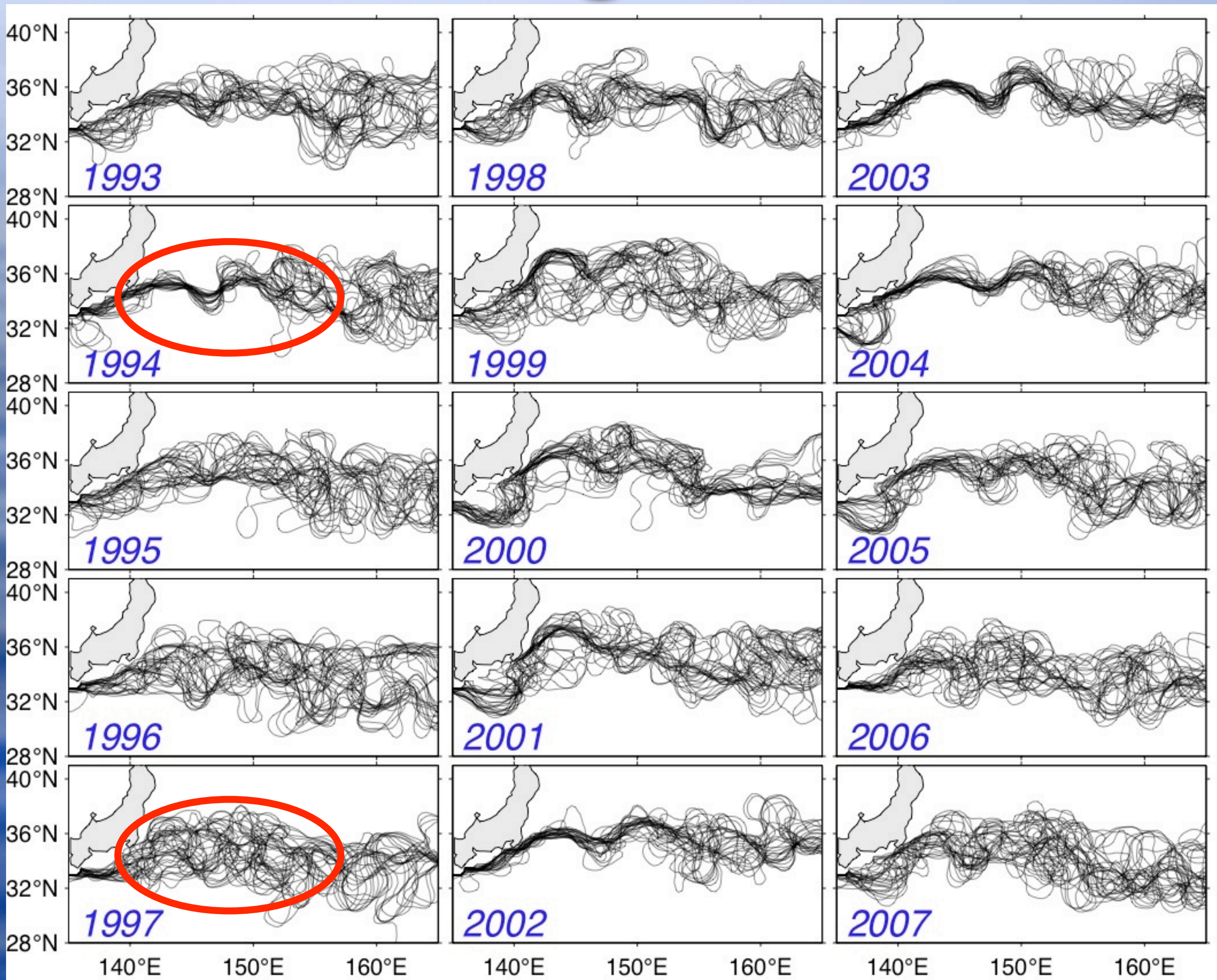


Based on SSTs, from satellite IR data

Kuroshio Extension (KE) Path Changes

Monthly paths from altimeter:

Stable vs. unstable periods



Qiu & Chen (*Deep-Sea Res.*, 2009)

Transitions Between Blocked and Zonal Flows
in a Barotropic Rotating Annulus with Topography

Zonal Flow
13–22 Dec. 1978

Blocked Flow
10–19 Jan. 1963

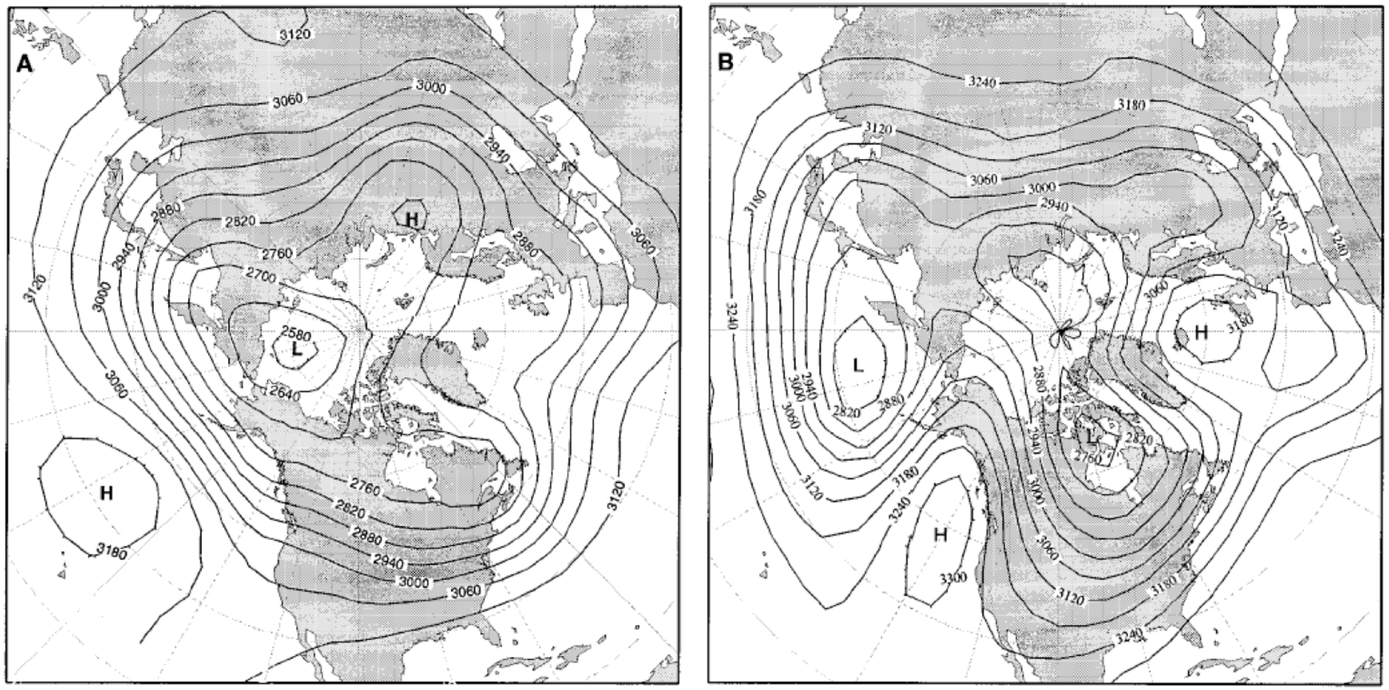


Fig. 1. Atmospheric pictures of (**A**) zonal and (**B**) blocked flow, showing contour plots of the height (m) of the 700-hPa (700 mbar) surface, with a contour interval of 60 m for both panels. The plots were obtained by averaging 10 days of twice-daily data for (A) 13 to 22 December 1978 and (B) 10 to 19 January 1963; the data are from the National Oceanic and Atmospheric Administration's Climate Analysis Center. The nearly zonal flow of (A) includes quasi-stationary, small-amplitude waves (32). Blocked flow advects cold Arctic air southward over eastern North America or Europe, while decreasing precipitation in the continent's western part (26).

Weeks, Tian, Urbach, Ide, Swinney, & Ghil (*Science*, 1997)

"Limited-contour" analysis for atmospheric low-frequency variability

10-day sequences of
subtropical jet paths:
blocked vs. zonal
flow regimes

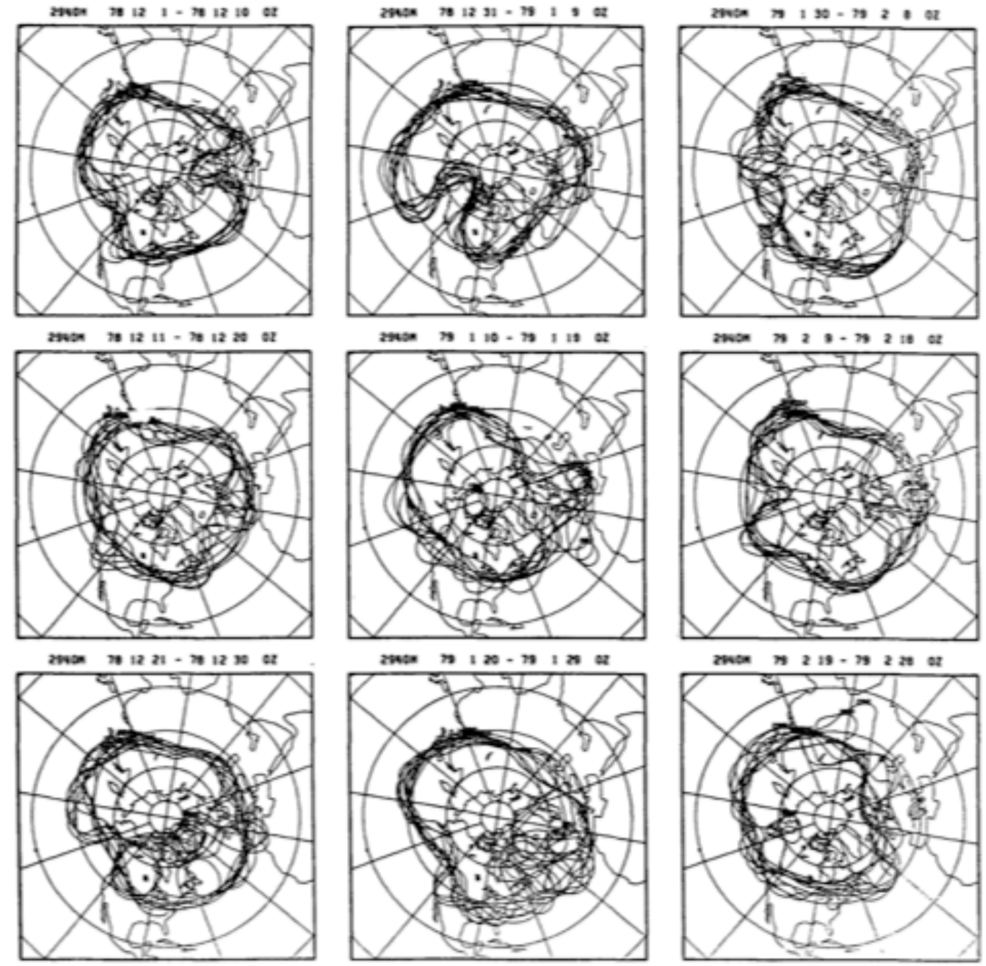


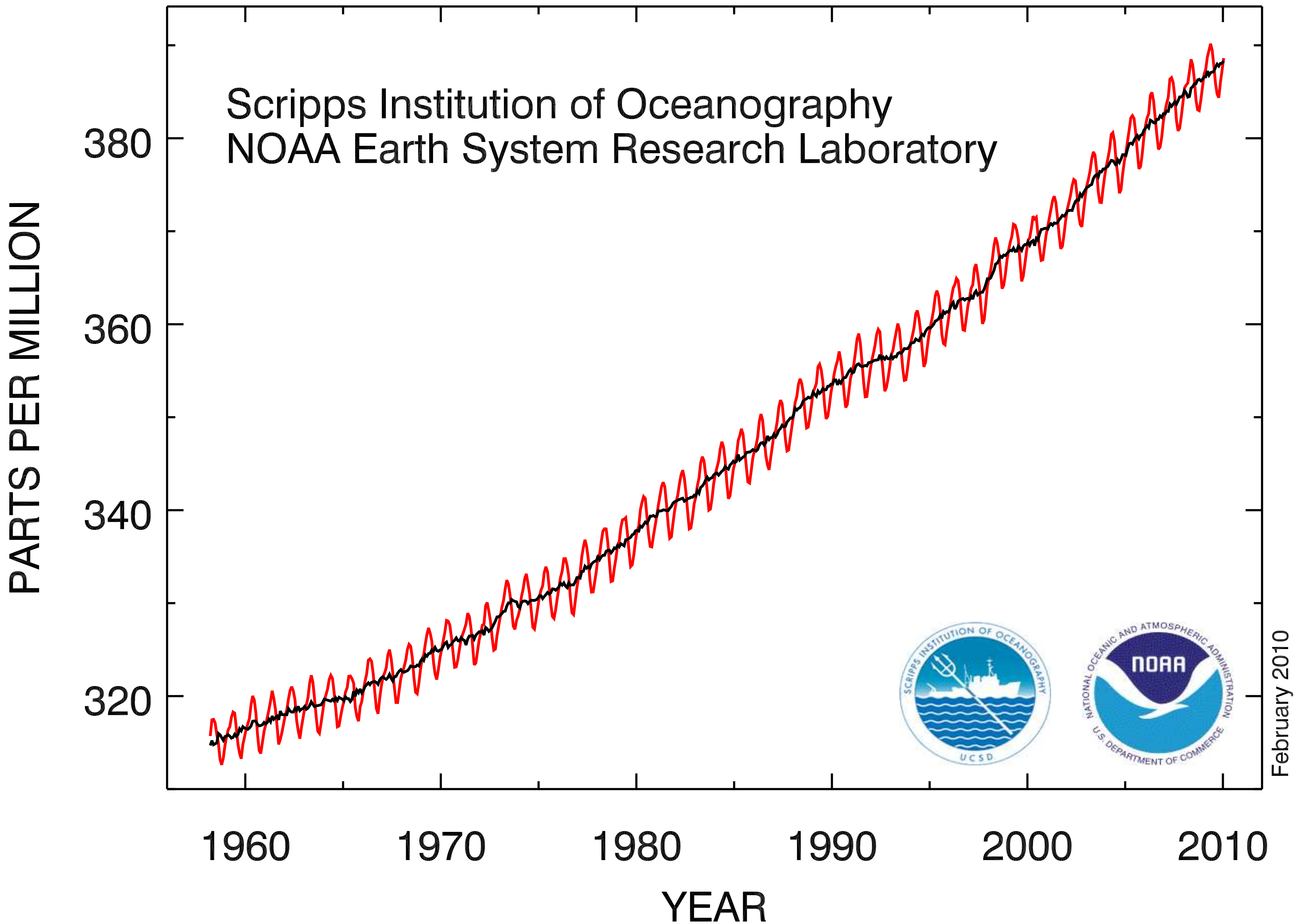
FIG. 1. Limited contour analysis of Northern Hemisphere (NH) flows. Daily contours of a prescribed height (2940 m in this case—roughly corresponding to the jet axis) are superimposed for successive 10-day intervals during NH winter 1978/79. Persistence is illustrated by some of the panels (see text for details).

Kimoto & Ghil, *JAS*, 1993a

Outline – Unsteady Flows & Climate

- Atmospheric & oceanic flows
  - scales of motion, in time & space
  - one person's signal ("**deterministic**") is another one's noise (**"stochastic"**)
- Time-dependent forcing
  - **intrinsic** vs. **forced** variability
  - **pullback** and **random** attractors
- An illustrative example
  - the Lorenz convection model with time-dependent forcing
- A "grand unification"
  - a mathematical definition of **climate sensitivity**
- Conclusions and references
  - what **do we** & **don't we** know?
  - selected bibliography

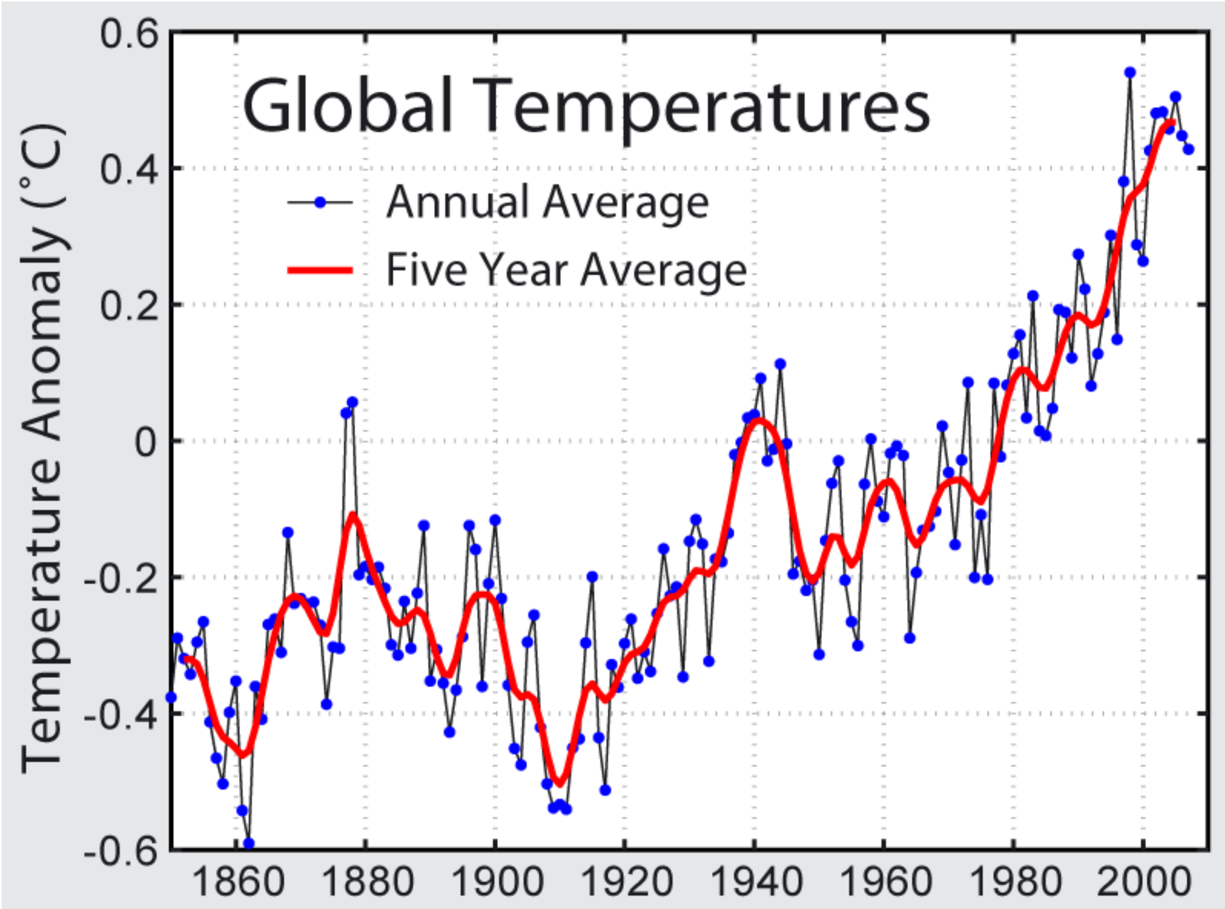
Atmospheric CO_2 at Mauna Loa Observatory



Temperatures and GHGs

Greenhouse gases (GHGs) go up,
temperatures go up:

It's gotta do with us, at least a bit,
doesn't it?



Wikicommons, from
Hansen *et al*. (*PNAS*, 2006);
see also http://data.giss.nasa.gov/
gistemp/graphs/

Global warming and its socio-economic impacts

Temperatures rise:

- What about impacts?
- How to adapt?

The answer, my friend, is blowing in the wind, *i.e.*, it depends on the accuracy and reliability of the forecast …

Source : IPCC (2007), *AR4, WGI, SPM*

MULTI-MODEL AVERAGES AND ASSESSED RANGES FOR SURFACE WARMING

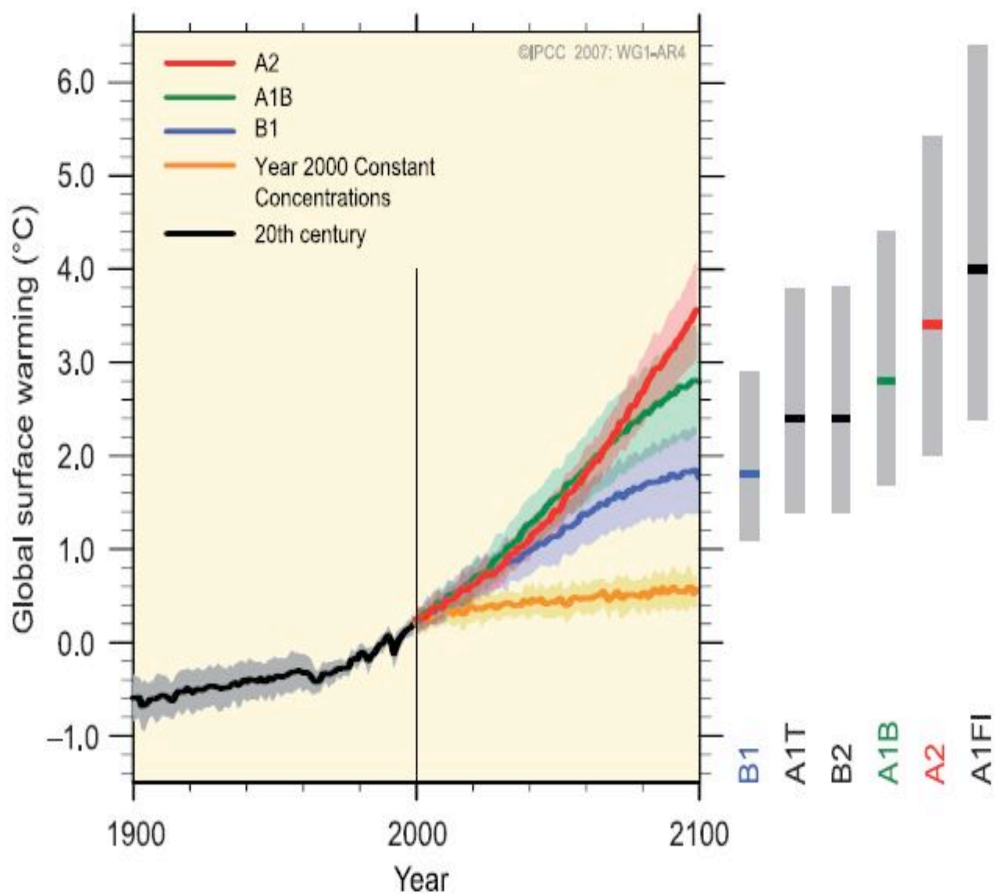


Figure SPM.5. *Solid lines are multi-model global averages of surface warming (relative to 1980–1999) for the scenarios A2, A1B and B1, shown as continuations of the 20th century simulations. Shading denotes the ±1 standard deviation range of individual model annual averages. The orange line is for the experiment where concentrations were held constant at year 2000 values. The grey bars at right indicate the best estimate (solid line within each bar) and the* **likely** *range assessed for the six SRES marker scenarios. The assessment of the best estimate and* **likely** *ranges in the grey bars includes the AOGCMs in the left part of the figure, as well as results from a hierarchy of independent models and observational constraints. {Figures 10.4 and 10.29}*

Time-Dependent Forcing –> Pullback Attractors

Consider the scalar, linear ordinary differential equation (ODE)

$$\dot{x} = -\alpha x + \sigma t\,, \quad \alpha > 0\,,\ \sigma > 0\,.$$

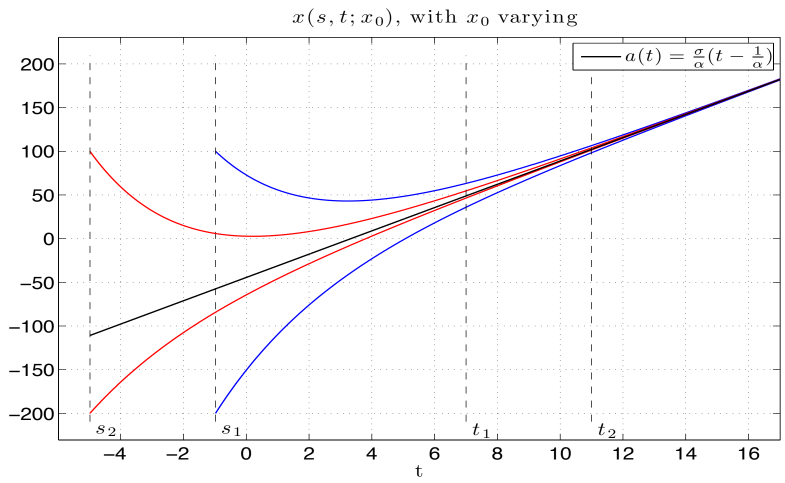
When there's no forcing, $\sigma = 0$, the ODE is purely dissipative

$$\dot{x} = -\alpha x\,,$$

and all solutions converge to the fixed point $x = 0$ as $t \to +\infty\,.$

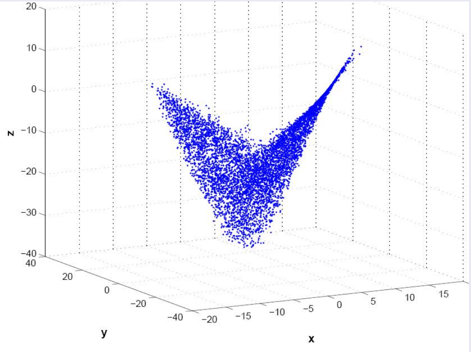
Now what about when we do have forcing, $\sigma \neq 0$?
At each time $t = t_1$, say, we have to "pull back" and start at some time $s = s_1 << t_1$, say, to see where the flow takes us at $t = t_1$.
As $s \to -\infty\,,$ we get the pullback attractor $a = a(t)$ in the figure,

$$a(t) = \frac{\sigma}{\alpha}\left(t - \frac{1}{\alpha}\right).$$



Random attractor of the stochastic Lorenz system

Snapshot of the random attractor (RA)



- A snapshot of the RA, $\mathcal{A}(\omega)$, computed at a fixed time t and for the same realization ω; it is made up of points transported by the stochastic flow, from the remote past $t - T$, $T >> 1$.
- We use multiplicative noise in the deterministic Lorenz model, with the classical parameter values $b = 8/3$, $\sigma = 10$, and $r = 28$.
- Even computed pathwise, this object supports meaningful statistics.

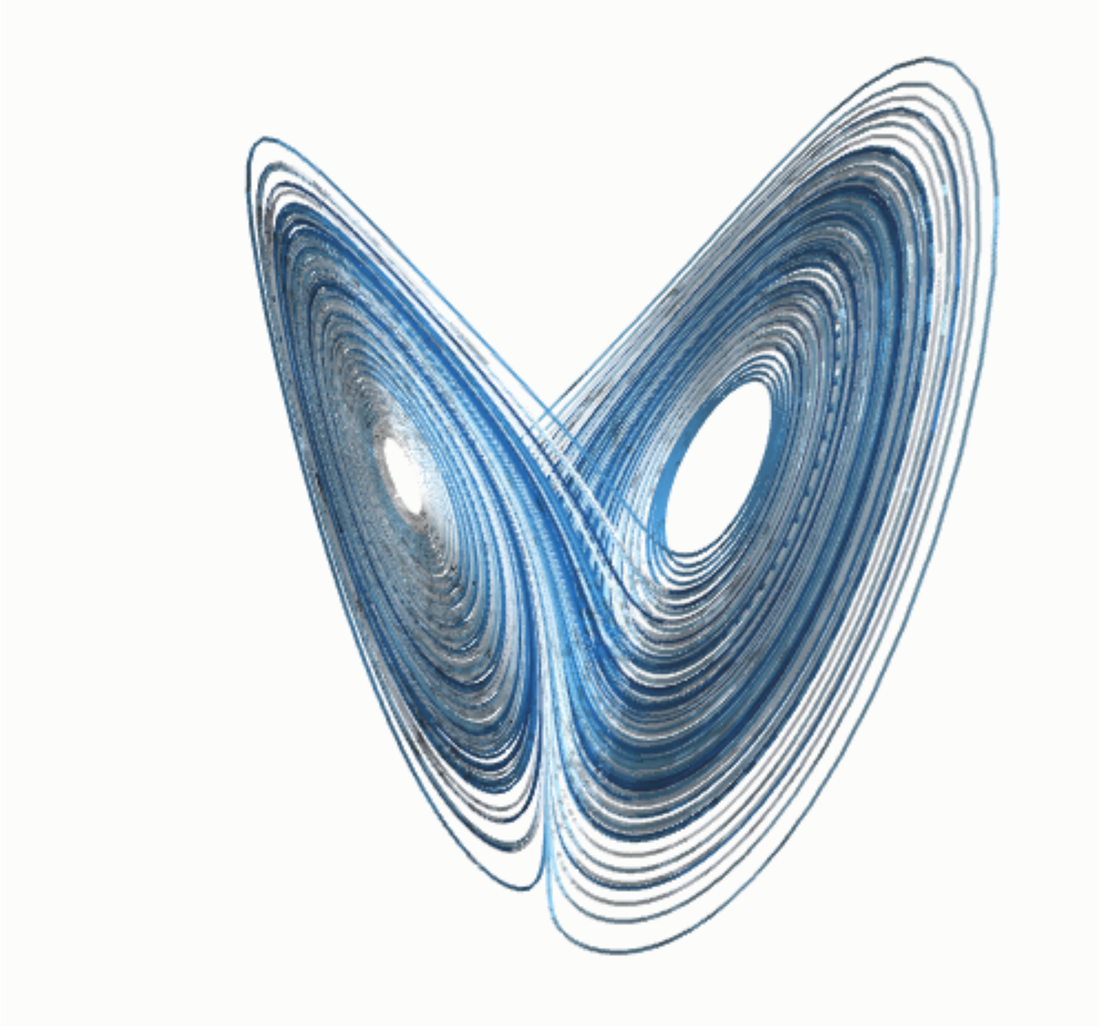
A day in the life of the Lorenz (1963) model's random attractor, or LORA for short;
see SI in Chekroun, Simonnet & Ghil (2011, *Physica D*)

Classical Strange Attractor

Physically closed system, modeled mathematically as autonomous system: neither deterministic (anthropogenic) nor random (natural) forcing.

The attractor is strange, but still fixed in time ~ “irrational” number.

Climate sensitivity ~ change in the average value (first moment) of the coordinates (x, y, z) as a parameter λ changes.



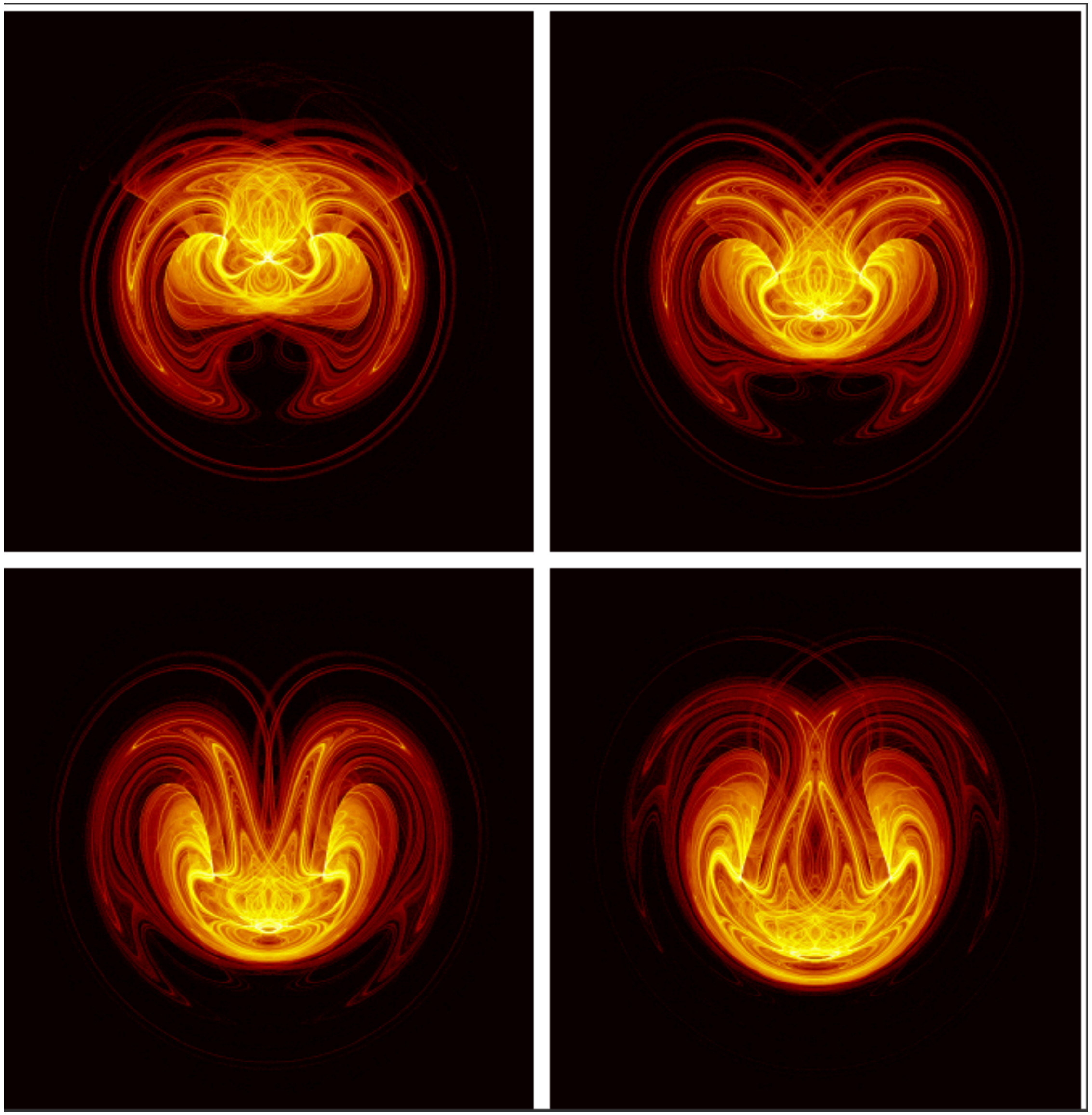
Random Attractor

Physically open system, modeled mathematically as non-autonomous system: allows for deterministic (anthropogenic) as well as random (natural) forcing.

The attractor is "pullback" and evolves in time ~ "imaginary" or "complex" number.

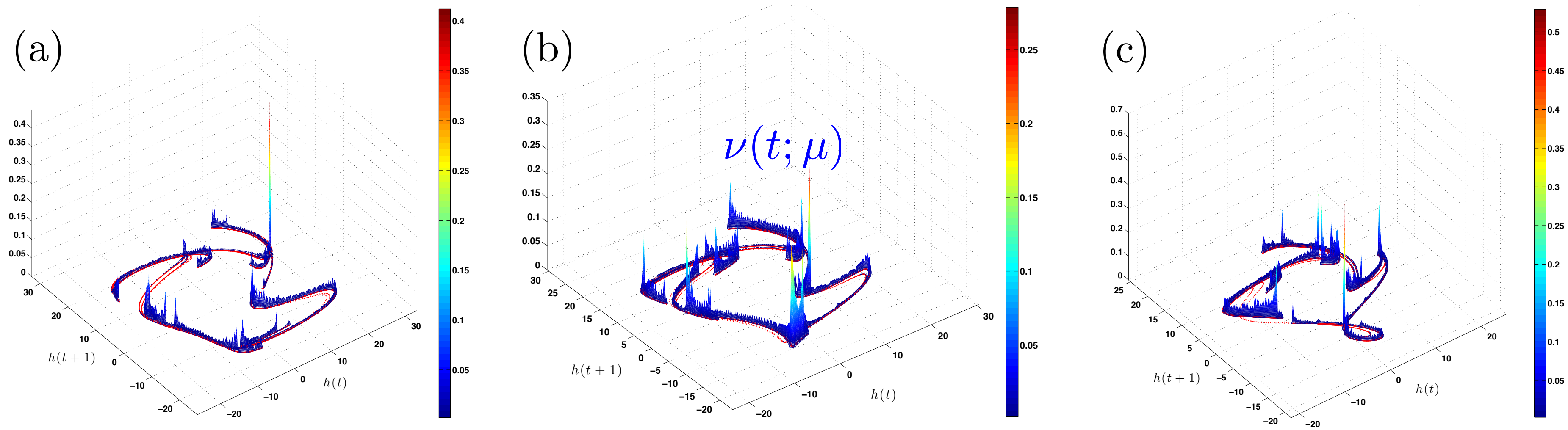
Climate sensitivity ~ change in the statistical properties (first and higher-order moments) of the attractor as one or more parameters (λ, μ, …) change.

Ghil (*Encyclopedia of Atmospheric Sciences*, 2nd ed., 2012)



How to define climate sensitivity or, What happens when there's natural variability?

This definition allows us to watch how "the earth moves," as it is affected by both natural and anthropogenic forcing, in the presence of natural variability, which includes both chaotic & random behavior:
chaotic + random behavior:



Clearly the invariant measure $\nu(t;\mu)$ changes in its position (i.e., its support), as well as in its probability density — with time t, as shown here — but also with respect to an arbitrary parameter μ, where $\mu = \tau$ in the present case. Hence, in general,

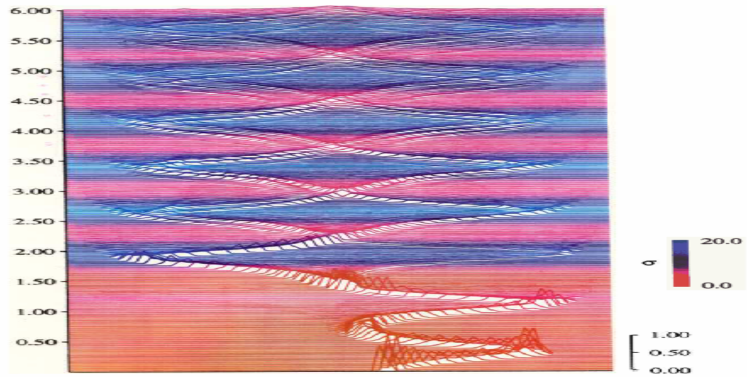
$$\gamma = \partial d_{\mathrm{W}} / \partial \mu.$$

Yet another (grand?) unification

Lorenz (*JAS*, 1963)

Climate is deterministic and autonomous,
but highly nonlinear.

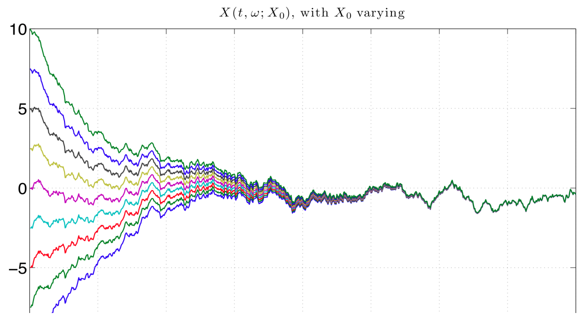
Trajectories diverge exponentially,
forward asymptotic PDF is multimodal.



Hasselmann (*Tellus*, 1976)

Climate is stochastic and noise-driven,
but quite linear.

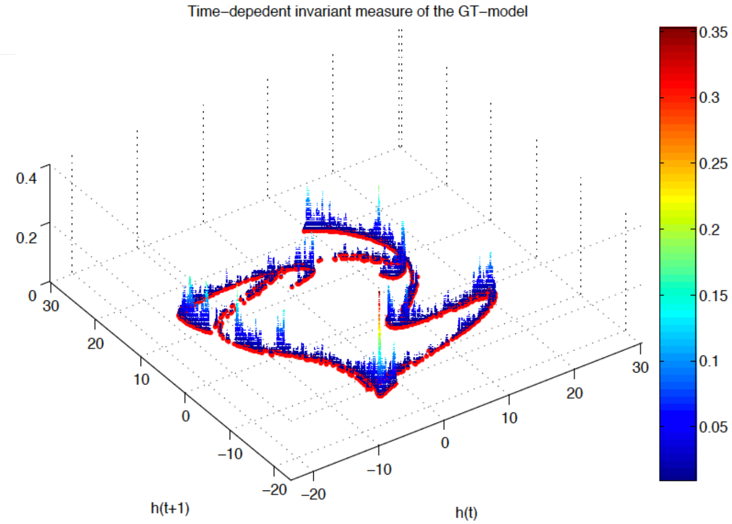
Trajectories decay back to the mean,
forward asymptotic PDF is unimodal.



Grand unification (?)

**Climate is deterministic + stochastic,
as well as highly nonlinear.**

**Internal variability and forcing interact
strongly, change and sensitivity
refer to both mean and higher moments.**



Concluding remarks –

What do we & don't we know?

What do we know?

- It's getting warmer.
- We do contribute to it.
- So we should act as best we know and can!

What do we know less well?

- By how much?
  - Is it getting warmer …
  - Do we contribute to it …
  - How does the climate system (atmosphere, ocean, ice, etc.) really work?
  - How does natural variability interact with anthropogenic forcing?

What to do?

- Better understand the system and its forcings.
- Explore the models', and the system's
  - robustness and sensitivity
  - pullback & random attractors

Some general references

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Math for Impacts - MAC2I

INdAM, La Sapienza, 13 March 2017

Business Cycles and the Impact of Major Natural Hazards

Michael Ghil (ENS, Paris, & UCLA)

with C. Colon & G. Weisbuch (ENS), B. Coluzzi (Roma), A. Groth (UCLA)
P. Dumas, S. Hallegatte (+World Bank) & J.-Ch. Hourcade (CIRED),
L. Sella, P. Terna & G. Vivaldo (U. of Torino)

Please visit these sites for more info.

https://dept.atmos.ucla.edu/tcd/

http://www.environnement.ens.fr/

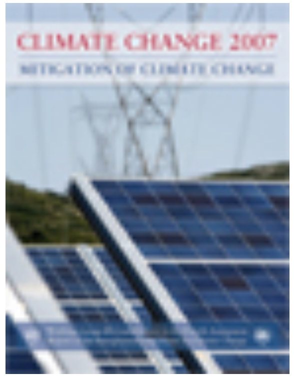
Motivation

- Coupled climate and socio-economic modeling
- Coordinating EU project on extreme events
    - in the geosciences and the socio-economic sciences
- Novel tools for both data analysis and modeling
    - SSA-MTM Toolkit for time series analysis
    - key tools for nonlinear and random dynamics
    - combined modeling and data studies

Motivation

- The IPCC process: Fourth Assessment Report (AR4)
- 3 working groups: various sources of uncertainties

  - Physical Science Basis
  - Impacts, Adaptation and Vulnerability
  - Mitigation of Climate Change



- Physical and socio-economic modeling

  - separate vs. coupled

- Ethics and policy issues

Outline

A. Endogenous business cycle (EnBC) model

- sawtooth-shaped business cycles, 5–6-year period
- impact of natural hazards
- vulnerability paradox ➔ fluctuation-dissipation relation

B. U.S. macroeconomic indicators

- methodology: singular-spectrum analysis (SSA) + multi-channel SSA (M-SSA)
- BEA data confirm the vulnerability paradox

C. EU & World data – work in progress

- Italy, Netherlands and UK data, correlations with USA
- 100 countries representing all economic regions
- commonalities and differences

D. Concluding remarks & bibliography

The need for models with endogenous dynamics

"The currently prevailing paradigm, namely that financial markets tend towards equilibrium, is both false and misleading; our current troubles can be largely attributed to the fact that the international financial system has been developed on the basis of that paradigm."

George Soros,
The New Paradigm for Financial Markets: The Credit Crisis of 2008 and What It Means,
BBS, PublicAffairs, New York, 2008

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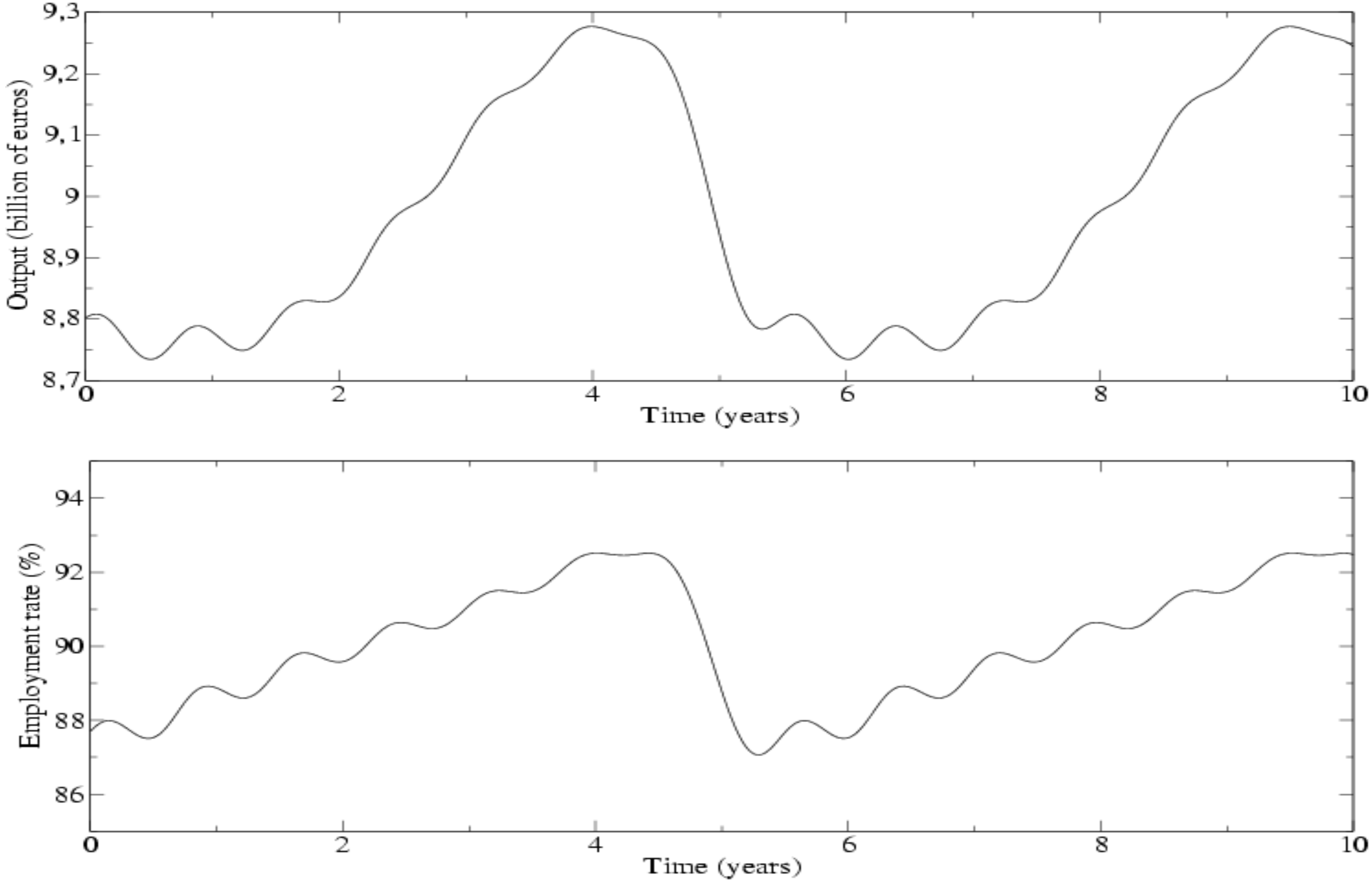
NEDyM (Non-equilibrium Dynamic Model)

- Represents an economy with one producer, one consumer, one goods that is used both to consume and invest.
- Based on the Solow (1956) model, in which all equilibrium constraints are replaced by dynamic relationships that involve adjustment delays.
- The NEDyM equilibrium is neo-classical and identical to that in the original Solow model. If the parameters are changing slowly, NEDyM has the same trajectories as the Solow model.
- Because of market adjustment delays, NEDyM model dynamics exhibits Keynesian features, with transient trajectory segments, in response to shocks.
- NEDyM possesses endogenous business cycles!

Hallegatte, Ghil, Dumas & Hourcade (*J. Econ. Behavior & Org.*, 2008)

Endogenous dynamics: an alternative explanation for business cycles



Endogenous business cycles (EnBCs) in NEDyM

- Business cycles originate from the profit–investment relationship
  (oscillations with a 5–6-year period) – Fukuyama (1989–92)?!

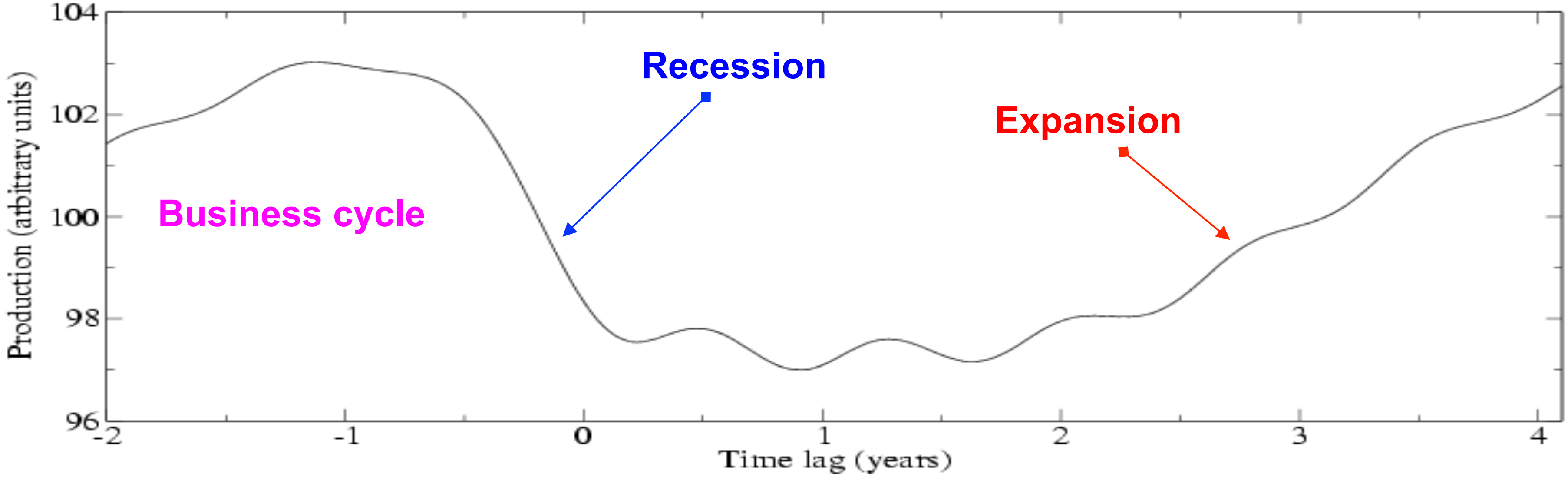
  higher profits => more investments => larger demand => higher profits

- Business cycles are limited in amplitude by three processes:
  - increase in labor costs when employment is high;
  - constraints in production and the consequent inflation in goods prices when demand increases too rapidly;
  - financial constraints on investment.

- EnBC models need to be calibrated and validated
  - harder than for real business cycle models (RBCs):
    fast and slow processes =>
    need a better definition of the business cycles =>
    study of BEA & NBER data!

Catastrophes and the state of the economy – I

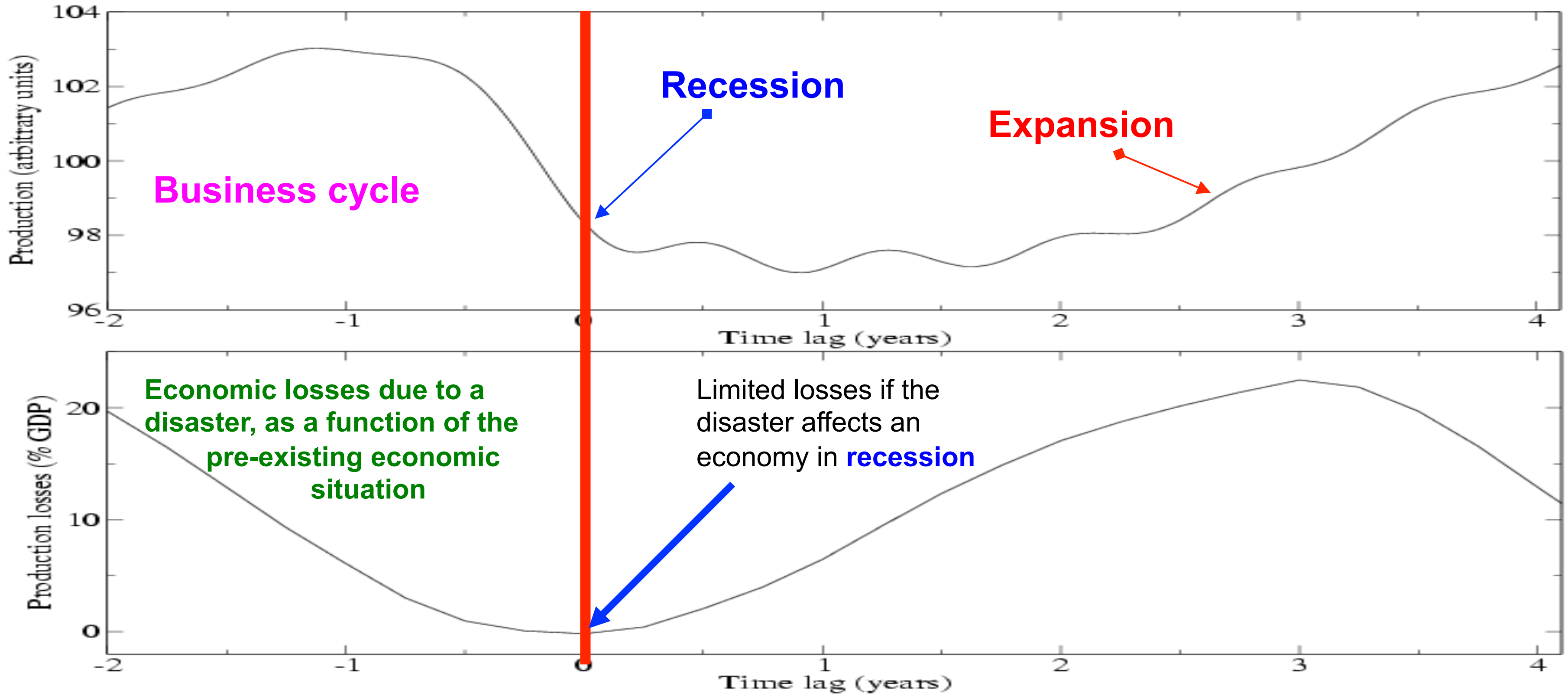
A vulnerability paradox: When does a disaster cause greater **long-term damage** to an economy, during its **expansion phase** or during a **recession**?



Hallegatte & Ghil, 2008, *Ecol. Econ.*, **68**, 582–592, doi:10.1016/j.ecolecon.2008.05.022

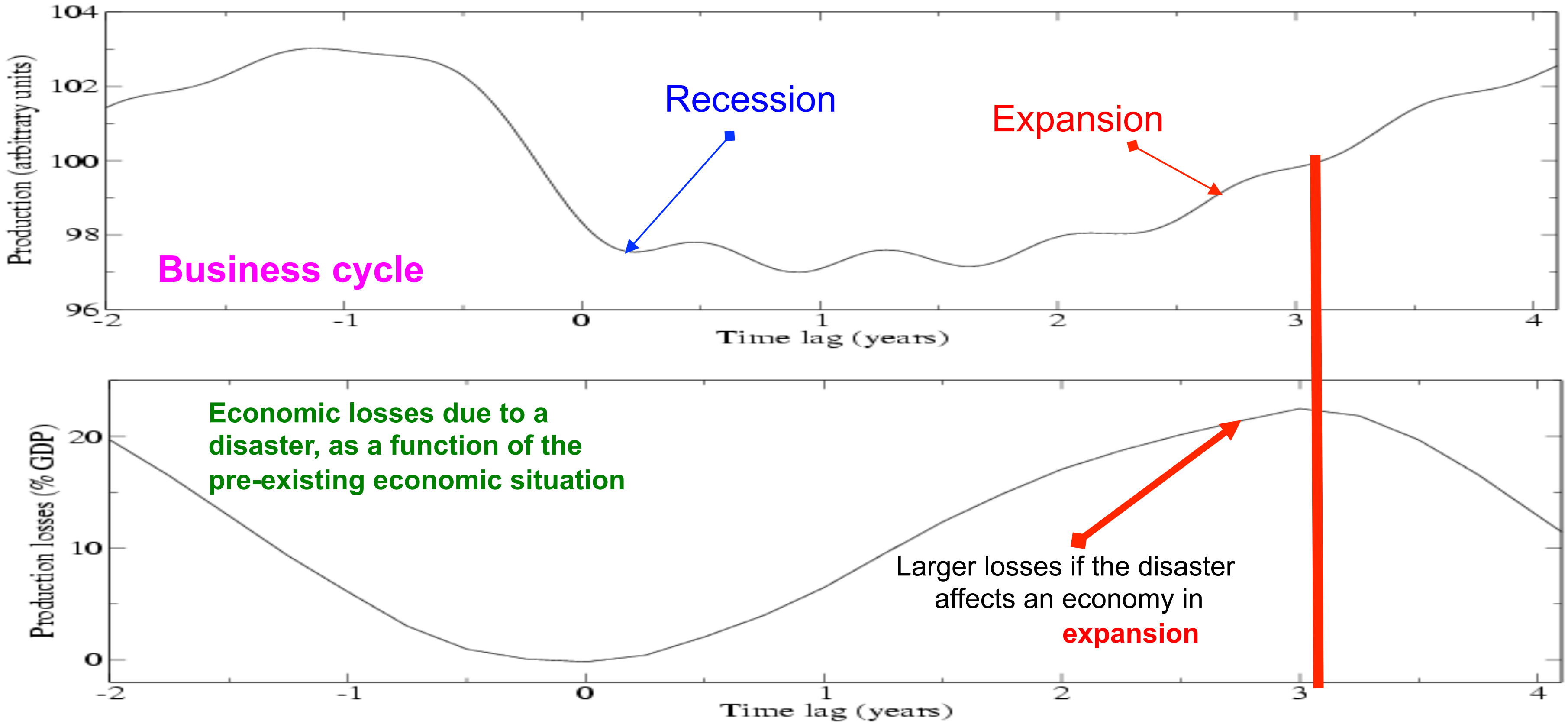
Catastrophes and the state of the economy – II

A vulnerability paradox:
A disaster that affects an economy during its **recession phase**…



Catastrophes and the state of the economy – III

… causes **fewer** long-term damages
than if it occurs during an **expansion!**



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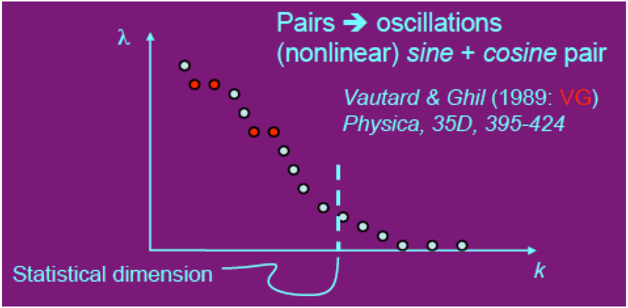
Singular Spectrum Analysis (SSA) – I

Spatial EOFs (PCA)	Temporal EOFs (SSA)
Expansion	
$\mathbf{\Phi}(t,x) = \sum_k \mathbf{a}_k(t)\,\mathbf{e}_k(x)$	$\mathbf{X}(t,s) = \sum_k \mathbf{a}_k(t)\,\mathbf{e}_k(s)$
Covariance	
$\mathbf{C}_\Phi(x,y) = \left\langle \mathbf{\Phi}(t,x)\mathbf{\Phi}(t,y)\right\rangle_t$	$\mathbf{C}_X(s,u) = \left\langle \mathbf{X}(t)\mathbf{X}(t+\lvert s-u\rvert)\right\rangle_t$
Eigendecomposition	
$\mathbf{C}_\Phi\,\mathbf{e}_k = \lambda_k\,\mathbf{e}_k$	$\mathbf{C}_X\,\mathbf{e}_k = \lambda_k\,\mathbf{e}_k$
Eigenelements	
$\mathbf{e}_k(x)$ $\quad x$ – space	$\mathbf{e}_k(s)$ $\quad s$ – time lag
	λ_k pairs $\to$ oscillations
	(nonlinear) sine + cosine pair

- Colebrook (1978); Weare & Nasstrom (1982); Broomhead & King (1986; BK); Fraedrich (1986); Vautard & Ghil (1989; VG).
- BK + VG: Analogy between Mañé-Takens embedding and the Wiener-Khinchin theorem.

Singular Spectrum Analysis (SSA) – II



- Truncation of the expansion to the S leading EOFs
  $\Rightarrow$ data-adaptive filter.
- Nearly equal eigenvalues $\Rightarrow$ nonlinear, anharmonic oscillation.

Stylized Facts of a Business Cycle – I

Need a more objective, quantitative description of the "typical business cycle." To do so we use two complementary approaches:

1. synchronization methods from dynamical systems ("chaos"); and
2. Advanced methods of time series analysis (SSA and M-SSA)

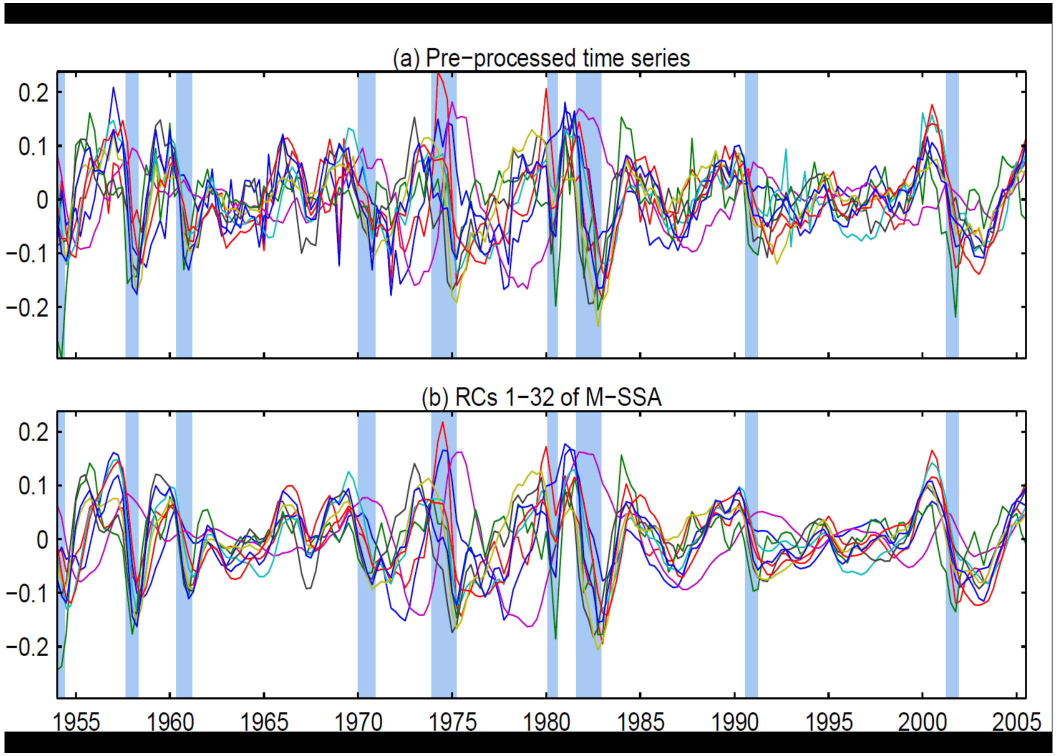
Bureau of Economic Analysis, www.bea.gov; 1947–2005.

9 variables:

gross domestic product (GDP), investment, consumption, employment rate (in %), price, total wage, imports, exports, and change in private inventories.

Groth, Ghil, Hallegatte and Dumas, submitted

Raw data, detrended and standardized



9-channel SSA ($D = 9$, $M = 24$ quarters)

Adaptive filtering, via multichannel singular-spectrum analysis (M-SSA);
vertical shaded bars are NBER-defined recessions

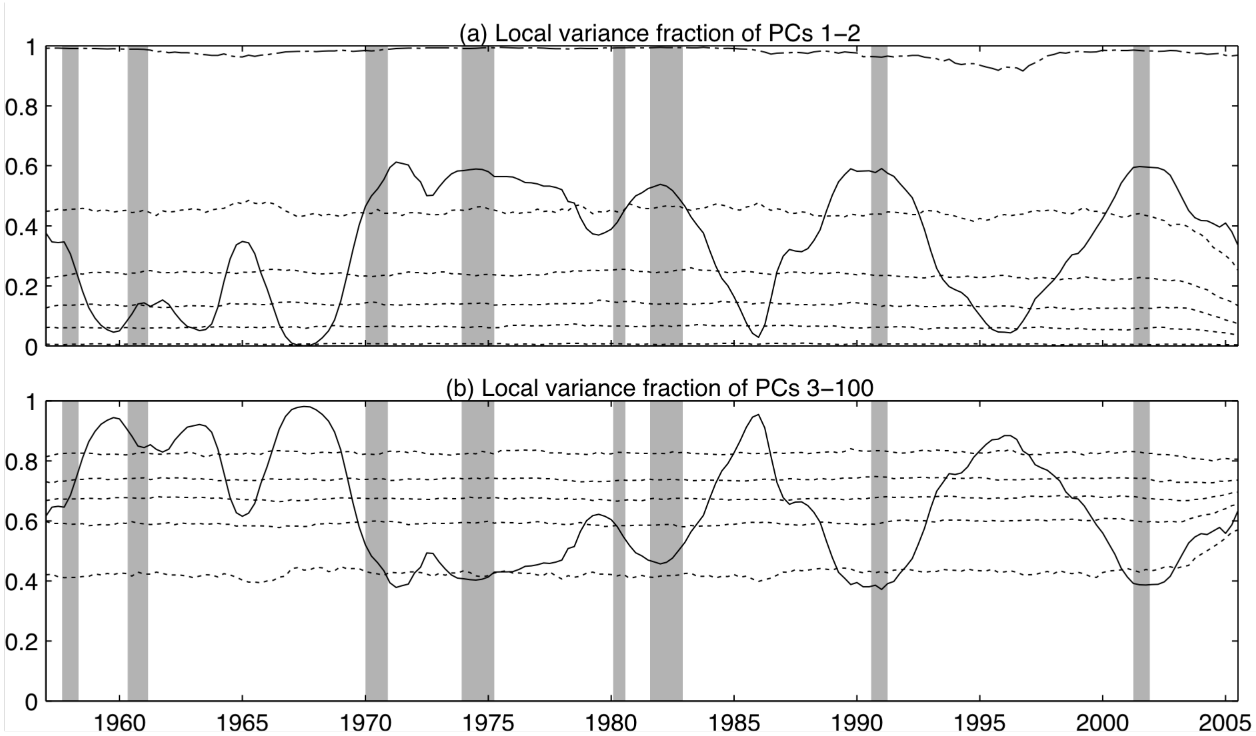
Stylized Facts of a Business Cycle – III

Consider the local variance fraction $V_{\mathcal{K}}(t)$ with $D = 9$, $M = 100$, and $A_k(t)$ the PCs:

$$V_{\mathcal{K}}(t) = \frac{\frac{\sum_{k \in \mathcal{K}} A_k(t)^2}{DM}}{\sum_{k=1} A_k(t)^2}$$

The "signal" fraction is largest during the recessions

The "noise" fraction is largest during the expansions



Vertical shaded bars are NBER-defined recessions

Groth, Ghil, Hallegatte and Dumas, submitted

Conclusions and outlook:
a hierarchy of economic models and data analysis methods

1. The highly idealized, **aggregate** **NEDyM model** exhibits fairly realistic, **endogenous business cycles (EnBCs):** period = 5–6 yr, sawtooth shape, good phasing of indices.
2. NEDyM displays a **vulnerability paradox**:
   - extreme-event consequences depend on the state of the economy;
   - they are more severe during an expansion than a recession.
3. This paradox is supported by
   - consequences of **Izmit (Marmara) earthquake, 1999**;
   - reconstruction process after the **2004 and 2005 hurricane seasons in Florida**.
4. **Better, quantitative characterization of business cycles**: spectral methods (**SSA & M-SSA**) + **phase synchronization; U.S. & EU done**, **World in process** (A. Groth, L. Sella, G. Vivaldo).
5. **U.S. economic data** (BEA, 1954–2005) support **a nonlinear fluctuation-dissipation theorem (FDT)** à la Ruelle.
6. Need more detailed, regional and sectorial models: C. Colon, B. Coluzzi, M. G., S.H., and G. Weisbuch have used simplified, **Boolean models** to study the economy as a **network of businesses** (suppliers and clients, etc.).
7. Compare **aggregate models** with **agent-based models (ABMs)** and with the **data, US + EU + global.**

A few references

Colon, C., and M. Ghil, 2017: Economic networks: Heterogeneity-induced vulnerability and loss of synchronization, *Chaos*, submitted.

Coluzzi, B., M. Ghil, S. Hallegatte, and G. Weisbuch, 2010: Boolean delay equations on networks in economics and the geosciences, *Intl. J. Bif. & Chaos*, **21**, 3511–3548.

Ghil, M., *et al*., 2002: Advanced spectral methods for climatic time series, *Rev. Geophys*., **40**(1), pp. **3**.1–**3**.41, doi: 10.1029/2000RG000092.

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Hallegatte, S., M. Ghil, P. Dumas, and J.-C. Hourcade, 2008: Business cycles, bifurcations and chaos in a neo-classical model with investment dynamics, *J. Econ. Behavior & Organization,* **67**, 57–77, doi: 10.1016/j.jebo.2007.05.001.

Hallegatte, S., and M. Ghil, 2008: Natural disasters impacting a macroeconomic model with endogenous dynamics, *Ecological Economics*, **68**, 582–592.

Sella, L., G. Vivaldo, A. Groth, and M. Ghil, 2016: Economic cycles and their synchronization: a comparison of cyclic modes in three European countries, *J. Bus. Cycle Res*., **12**, 25–48.

The deeper motivations of economic modeling

"Really, Karl! Can't I mention the high price of kohlrabi without getting a manifesto?"

Reserve slides

Unfortunately, things aren't all that easy!

What to do?

Try to achieve better interpretation of, and agreement between, models …

Ghil, M., 2002: Natural climate variability, in *Encyclopedia of Global Environmental Change*, T. Munn (Ed.), Vol. 1, Wiley

Natural variability introduces additional complexity into the anthropogenic climate change problem

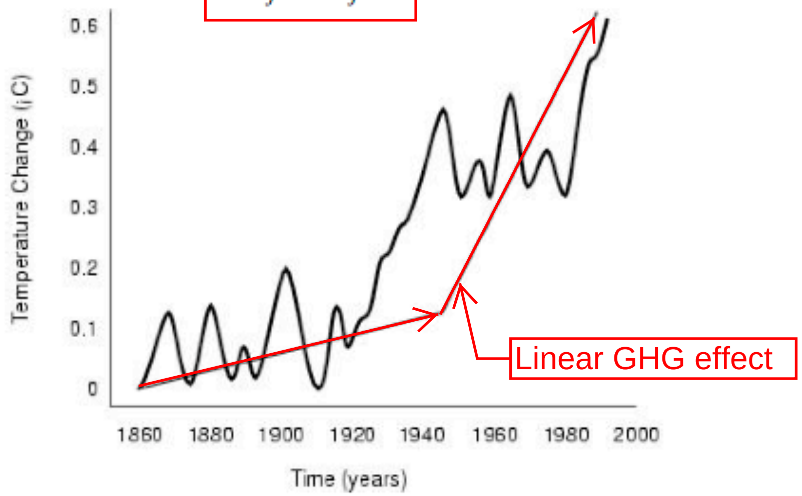
The most common interpretation of observations and GCM simulations of climate change is still in terms of a scalar, linear Ordinary Differential Equation (ODE)

$$\boxed{c\frac{dT}{dt} = -kT + Q}$$

$$k = \sum k_i \text{ – feedbacks (+ve and -ve)}$$

$$Q = \sum Q_j \text{ – sources \& sinks}$$

$$Q_j = Q_j(t)$$



Linear response to CO_2 vs. observed change in T

Hence, we need to consider instead a system of nonlinear Partial Differential Equations (PDEs), with parameters and multiplicative, as well as additive forcing (deterministic + stochastic)

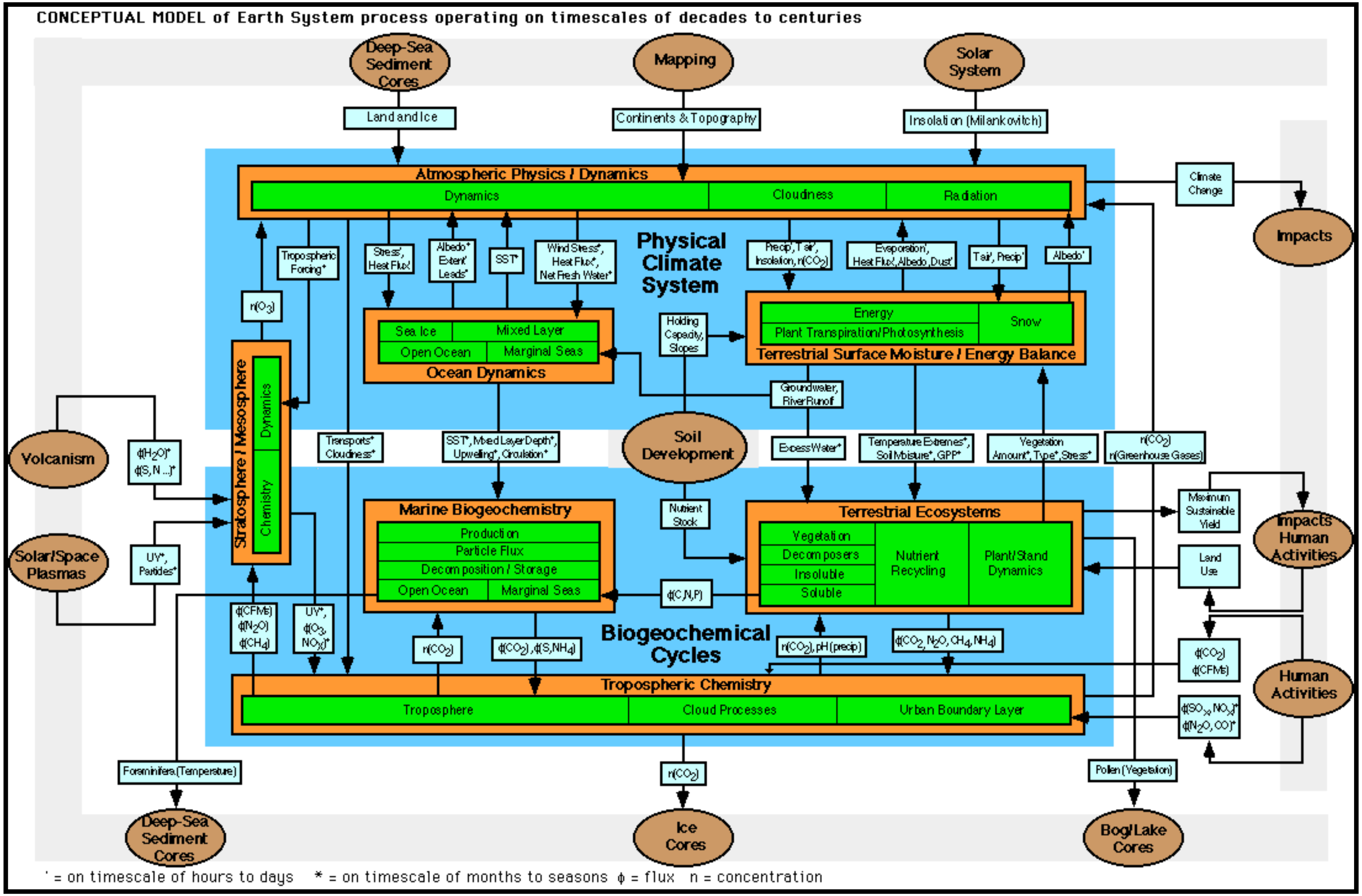
$$\boxed{\frac{d\mathbf{X}}{dt} = \mathbf{N}(\mathbf{X}, t, \mu, \beta)}$$

So what's it gonna be like, by 2100?

Table SPM.2. *Recent trends, assessment of human influence on the trend and projections for extreme weather events for which there is an observed late-20th century trend.* {Tables 3.7, 3.8, 9.4; Sections 3.8, 5.5, 9.7, 11.2–11.9}

Phenomenon[a] and direction of trend	Likelihood that trend occurred in late 20th century (typically post 1960)	Likelihood of a human contribution to observed trend[b]	Likelihood of future trends based on projections for 21st century using SRES scenarios
Warmer and fewer cold days and nights over most land areas	*Very likely*[c]	*Likely*[d]	*Virtually certain*[d]
Warmer and more frequent hot days and nights over most land areas	*Very likely*[e]	*Likely (nights)*[d]	*Virtually certain*[d]
Warm spells/heat waves. Frequency increases over most land areas	*Likely*	*More likely than not*[f]	*Very likely*
Heavy precipitation events. Frequency (or proportion of total rainfall from heavy falls) increases over most areas	*Likely*	*More likely than not*[f]	*Very likely*
Area affected by droughts increases	*Likely* in many regions since 1970s	*More likely than not*	*Likely*
Intense tropical cyclone activity increases	*Likely* in some regions since 1970	*More likely than not*[f]	*Likely*
Increased incidence of extreme high sea level (excludes tsunamis)[g]	*Likely*	*More likely than not*[f,h]	*Likely*[i]

F. Bretherton's "horrendogram" of Earth System Science



Earth System Science Overview, *NASA Advisory Council*, 1986

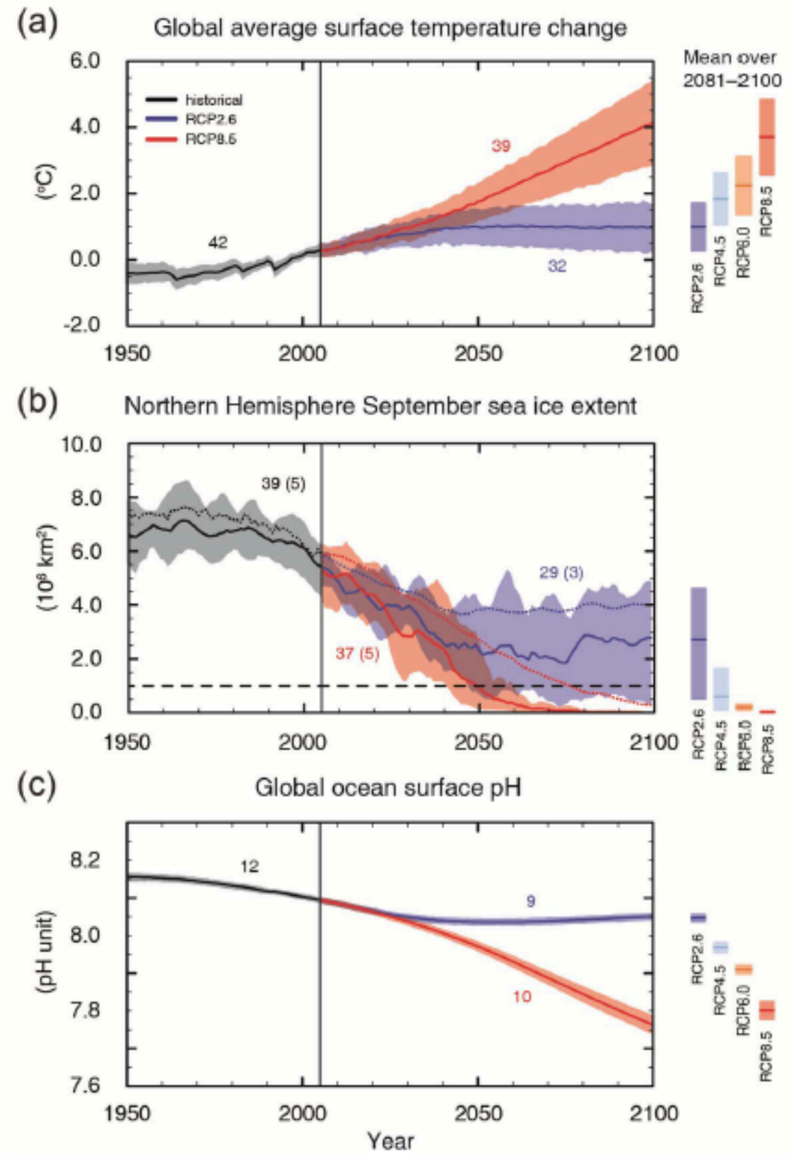
Global warming and its socio-economic impacts– II

Temperatures rise:

- What about impacts?
- How to adapt?

AR5 vs. AR4

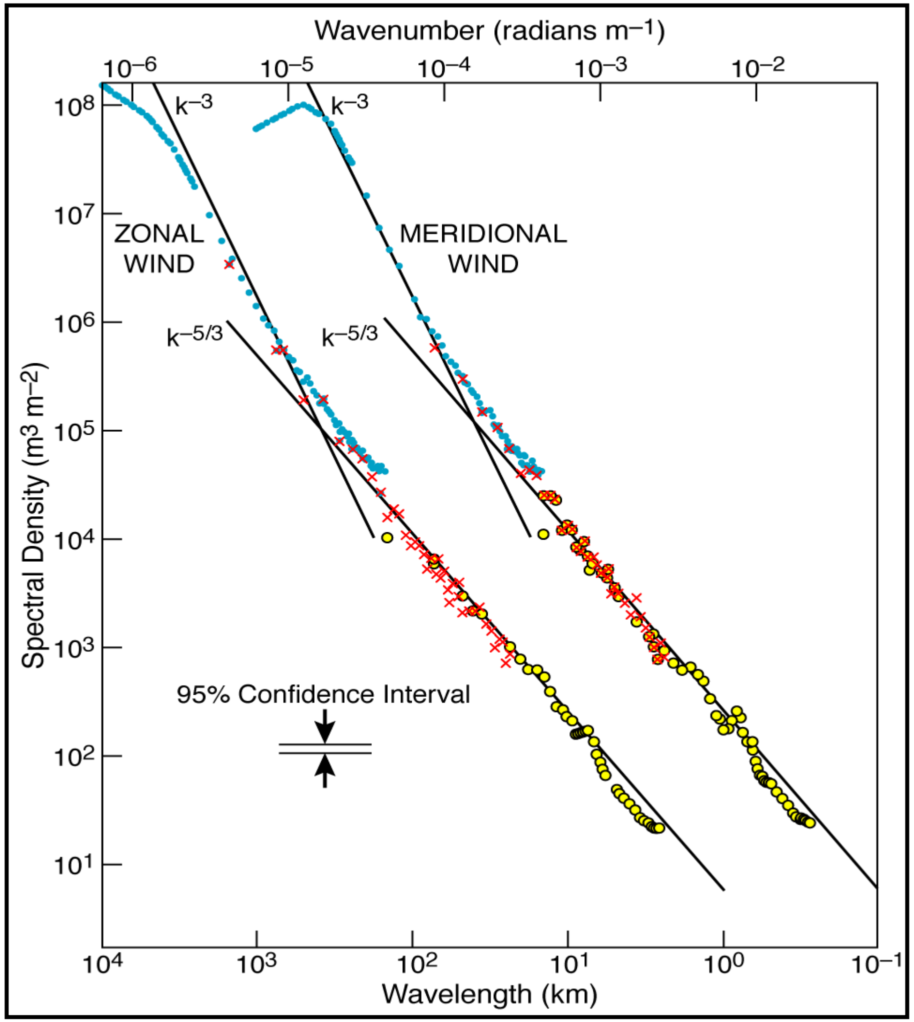
A certain air of *déjà vu*:
GHG "scenarios" have been replaced by "representative concentration pathways" (RCPs), more dire predictions, but the uncertainties remain.



Source : IPCC (2013), AR5, WGI, SPM

But deterministic chaos doesn't explain all: there are many other sources of irregularity!

- The energy spectrum of the atmosphere and ocean is "full": all space & time scales are active and they all contribute to forecasting uncertainties.
- Still, one can imagine that the longest & slowest scales contribute most to the longest-term forecasts.
- "One person's signal is another person's noise."

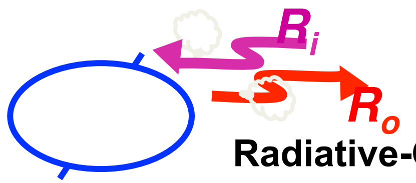


After Nastrom & Gage (*JAS*, 1985)

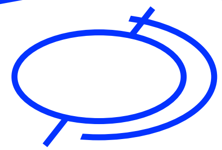
Climate models (atmospheric & coupled) : A classification

- *Temporal*
  - stationary, (quasi-)equilibrium
  - transient, climate variability

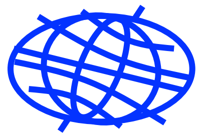
- *Space*
  - 0-D (dimension 0)
  - 1-D
    - vertical
    - latitudinal
  - 2-D
    - horizontal
    - meridional plane
  - 3-D, GCMs (General Circulation Model)
  - Simple and intermediate 2-D & 3-D models

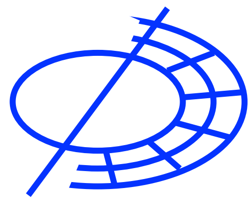


Radiative-Convective Model(*RCM*)



Energy Balance Model (*EBM*)





- *Coupling*
  - Partial
    - unidirectional
    - asynchronous, hybrid
  - Full

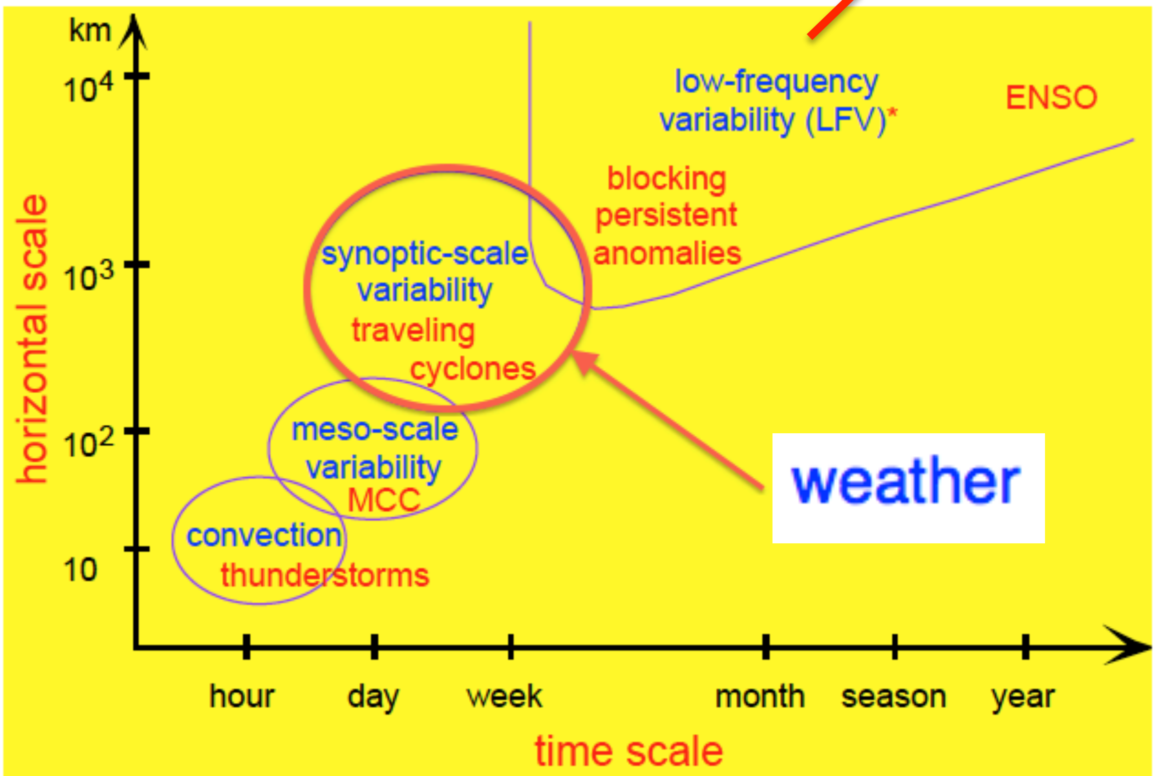
➔ ***Hierarchy***: back-and-forth between the simplest and the most elaborate model, and between the models and the observational data

Multiple scales of motion: Space-time organization

Observations

climate

- The most active scales lie along a diagonal in this space *vs.* time plot.
- Why this is so is far from clear as of now.
- We'll deal with weather first, then climate.



N.B. A **high-variability ridge** lies close to the **diagonal** of the plot (*cf.* also Fraedrich & Böttger, 1978, *JAS*)

* **LFV** ≅ 10–100 days (intraseasonal)

Can we, nonlinear dynamicists, help?

The uncertainties
might be ***intrinsic***,
rather than mere
"tuning problems"

If so, maybe
stochastic structural stability could help!

Might fit in nicely with
recent taste for
"stochastic
parameterizations"

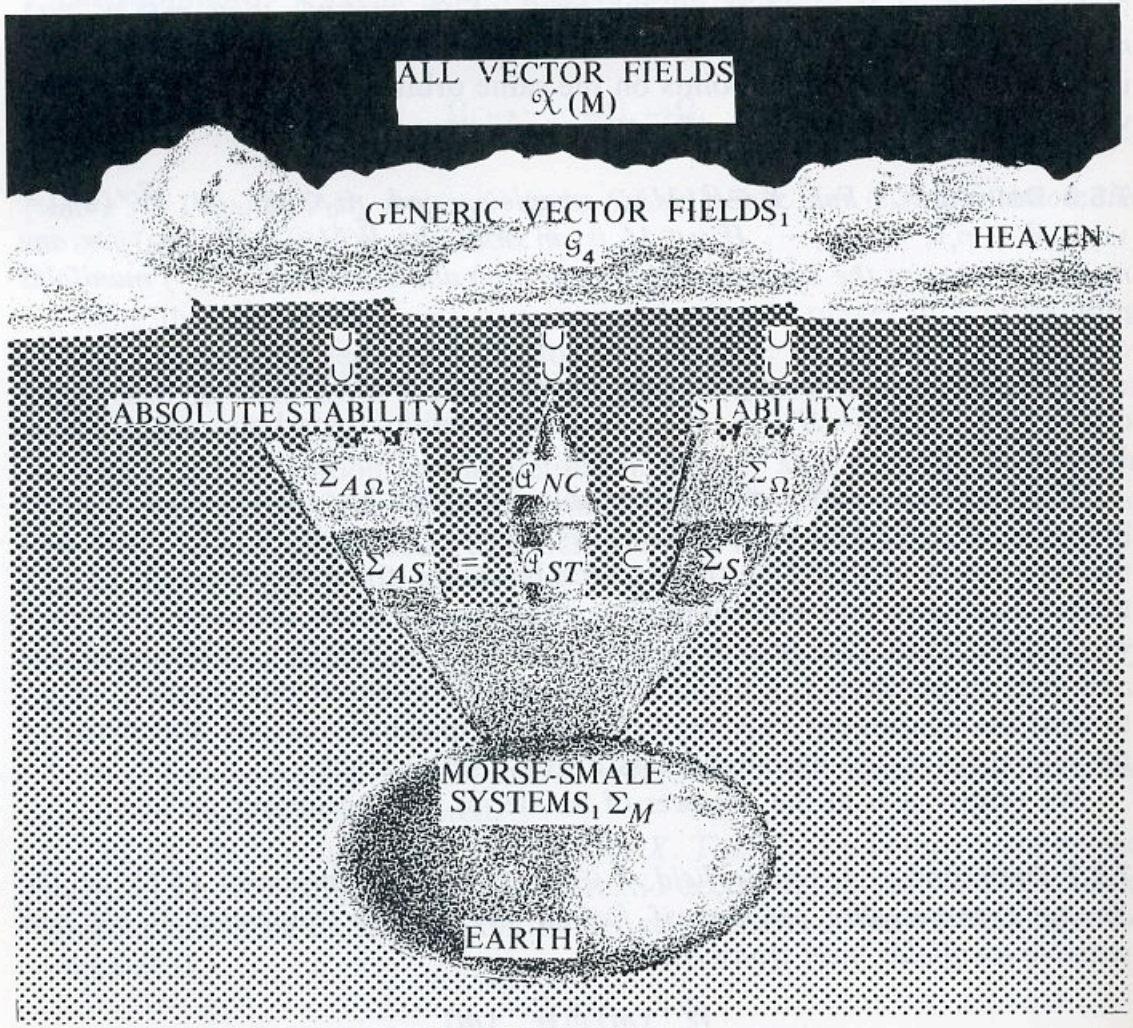


Figure 7.5-1. The three towers of differentiable dynamics.

The DDS dream of structural stability (from *Abraham & Marsden*, 1978)

This theory is the counterpart for randomly forced dynamical systems (RDS) of the *geometric theory* of ordinary differential equations (ODEs). It allows one to treat stochastic differential equations (SDEs) — and more general systems driven by noise — as flows in (phase space)$\times$(probability space).

SDE$\sim$ODE, RDS$\sim$DDS, L. Arnold (1998)$\sim$V.I. Arnol'd (1983).

Setting:

(i) A phase space X. **Example**: $\mathbb{R}^n$.

(ii) A probability space $(\Omega, \mathcal{F}, \mathbb{P})$. **Example**: The Wiener space $\Omega = C_0(\mathbb{R}; \mathbb{R}^n)$ with Wiener measure $\mathbb{P}$.

(iii) A model of the noise $\theta(t) : \Omega \to \Omega$ that preserves the measure $\mathbb{P}$, i.e. $\theta(t)\mathbb{P} = \mathbb{P}$; θ is called the driving system.
Example: $W(t, \theta(s)\omega) = W(t + s, \omega) - W(s, \omega)$;
it starts the noise at s instead of $t = 0$.

(iv) A mapping $\varphi : \mathbb{R} \times \Omega \times X \to X$ with the cocycle property.
Example: The solution operator of an SDE.

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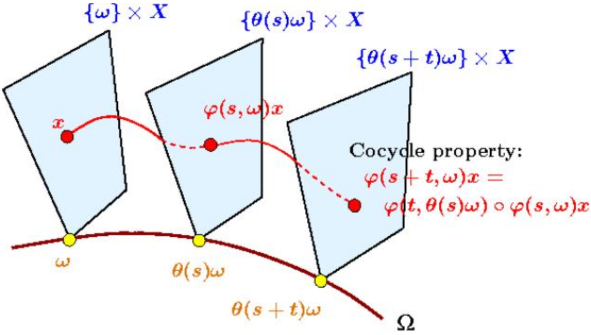
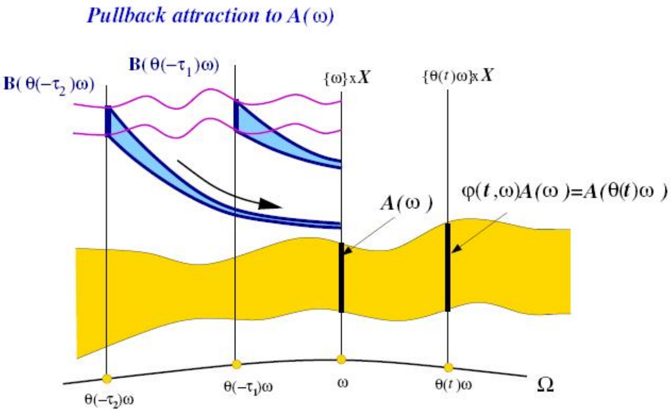


- φ **is a random dynamical system (RDS)**
- $\Theta(t)(x,\omega) = (\theta(t)\omega, \varphi(t,\omega)x)$ **is a flow on the bundle**

A random attractor $\mathcal{A}(\omega)$ is both *invariant* and "pullback" *attracting*:

(a) **Invariant**: $\varphi(t,\omega)\mathcal{A}(\omega) = \mathcal{A}(\theta(t)\omega)$.

(b) **Attracting**: $\forall B \subset X$, $\lim_{t\to\infty} \mathrm{dist}(\varphi(t,\theta(-t)\omega)B, \mathcal{A}(\omega)) = 0$ a.s.



Sample measures for an NDDE model of ENSO

The Galanti-Tziperman (GT) model (*JAS*, 1999)

$$\frac{dT}{dt} = -\epsilon_T T(t) - M_0(T(t) - T_{sub}(h(t))),$$

Neutral delay-differential equation (NDDE), derived from Cane-Zebiak and Jin-Neelin models for ENSO: T is East-basin SST and h is thermocline depth.

$$\begin{aligned} h(t) = {} & M_1 e^{-\epsilon_m(\tau_1+\tau_2)} h(t-\tau_1-\tau_2) \\ & -M_2\tau_1 e^{-\epsilon_m(\frac{\tau_1}{2}+\tau_2)} \mu(t-\tau_2-\frac{\tau_1}{2}) T(t-\tau_2-\frac{\tau_1}{2}) \\ & +M_3\tau_2 e^{-\epsilon_m\frac{\tau_2}{2}} \mu(t-\frac{\tau_2}{2}) T(t-\frac{\tau_2}{2}). \end{aligned}$$

Seasonal forcing given by $\mu(t) = 1 + \epsilon\cos(\omega t + \phi)$. The pullback attractor and its invariant measures were computed.

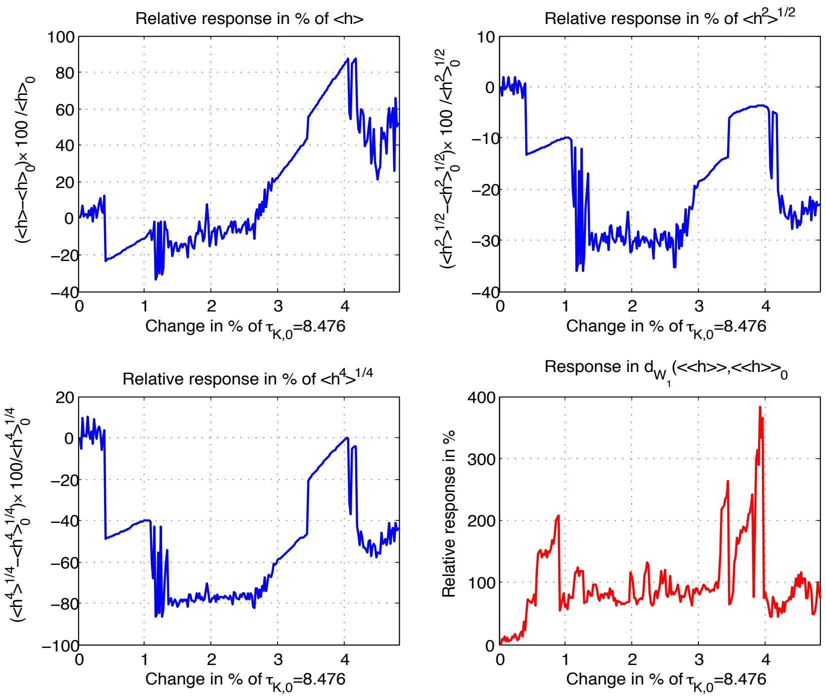


Figure shows the changes in the mean, 2nd & 4th moment of $h(t)$, along with the Wasserstein distance d_W, for changes of 0–5% in the delay parameter $\tau_{\kappa,0}$

Note intervals of both smooth & rough dependence!

letters to nature

Nature **350**, 324 - 327 (1991); doi:10.1038/350324a0

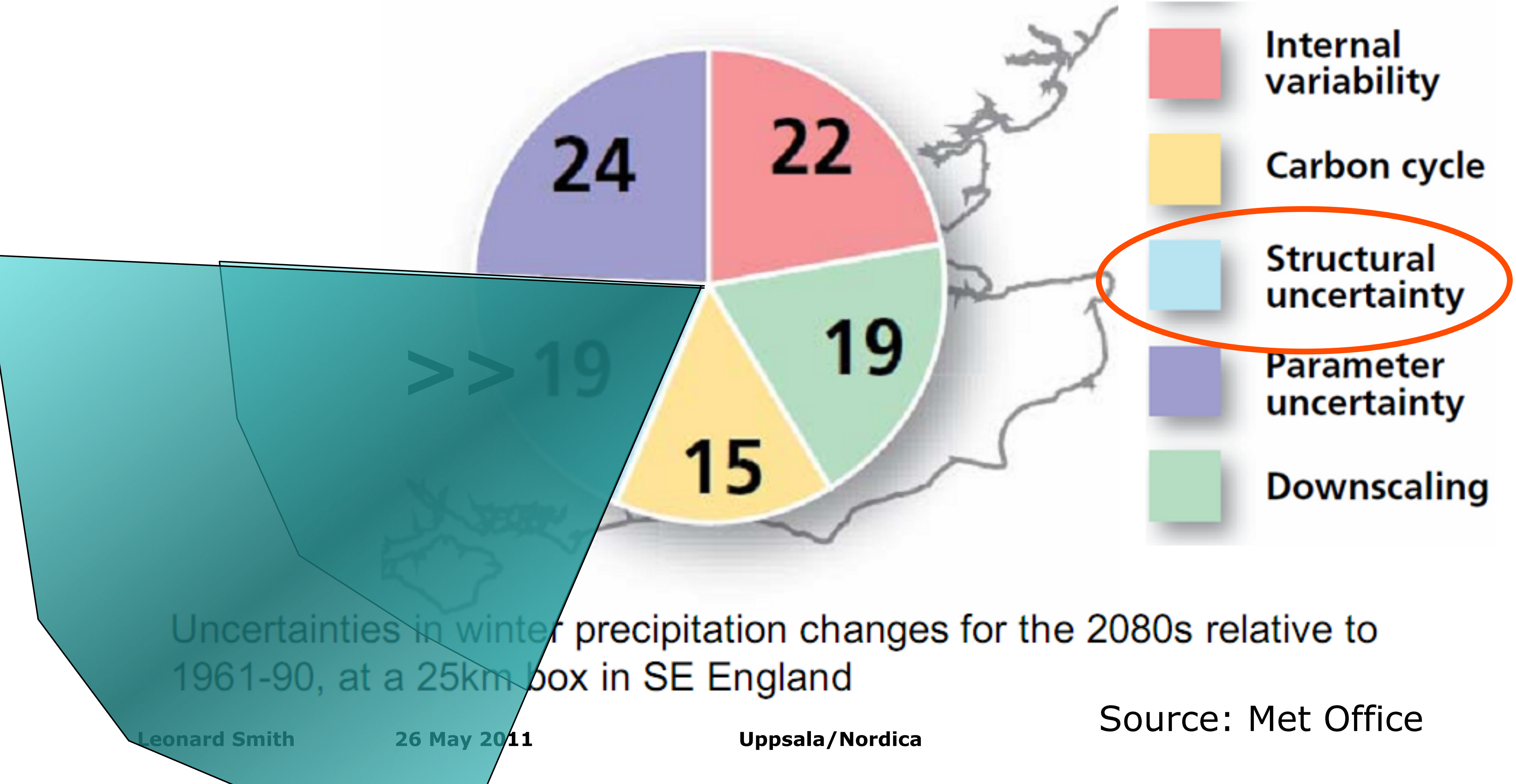
Interdecadal oscillations and the warming trend in global temperature time series

M. Ghil & R. Vautard

THE ability to distinguish a warming trend from natural variability is critical for an understanding of the climatic response to increasing greenhouse-gas concentrations. Here we use singular spectrum analysis[1] to analyse the time series of global surface air tem-peratures for the past 135 years[2], allowing a secular warming trend and a small number of oscillatory modes to be separated from the noise. The trend is flat until 1910, with an increase of 0.4 °C since then. The oscillations exhibit interdecadal periods of 21 and 16 years, and interannual periods of 6 and 5 years. The interannual oscillations are probably related to global aspects of the El Niño-Southern Oscillation (ENSO) phenomenon[3]. The interdecadal oscillations could be associated with changes in the extratropical ocean circulation[4]. The oscillatory components have combined (peak-to-peak) amplitudes of 0.2 °C, and therefore limit our ability to predict whether the inferred secular warming trend of 0.005 °Cyr^{-1} will continue. This could postpone incontrovertible detection of the greenhouse warming signal for one or two decades.

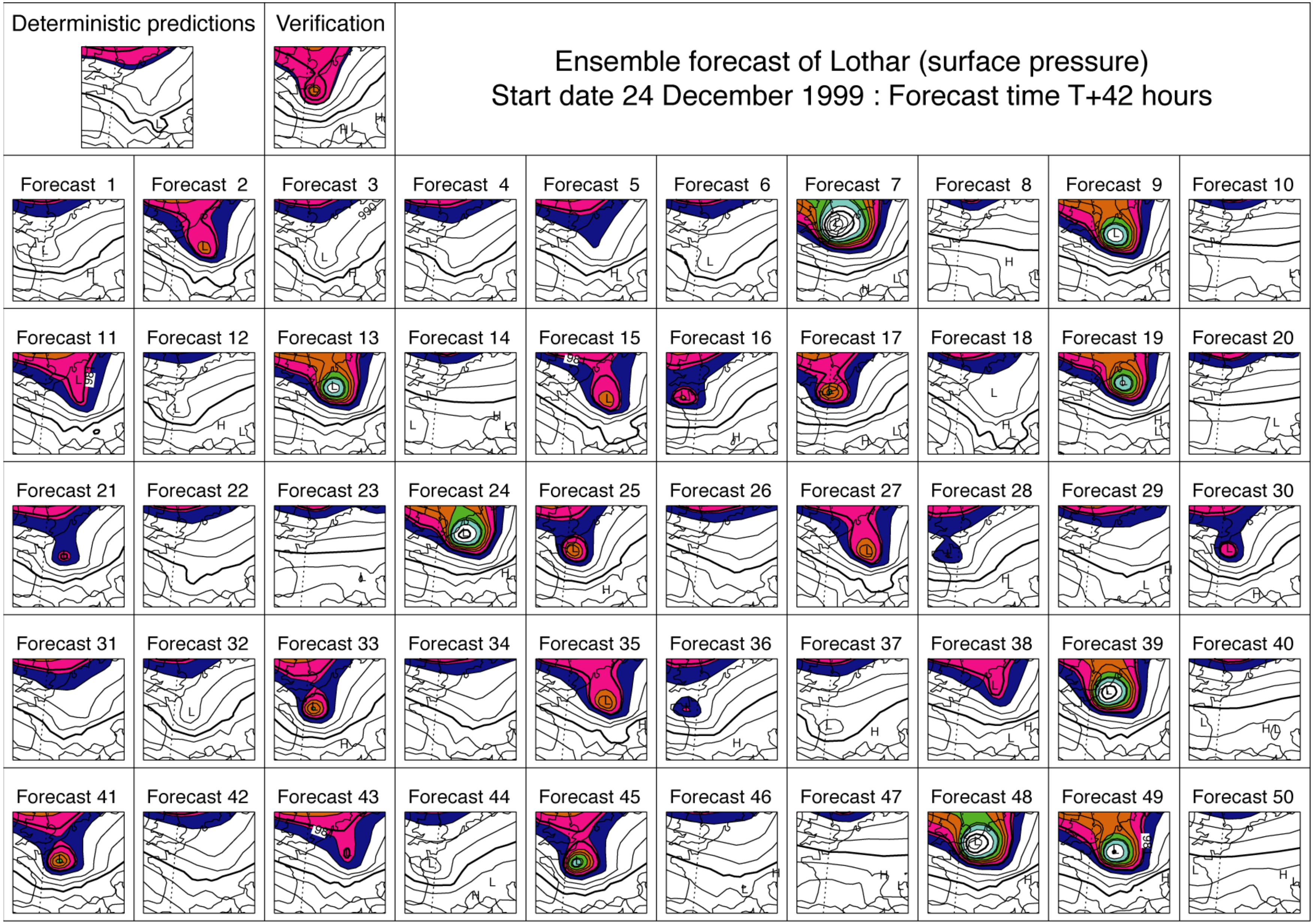
How important are different sources of uncertainty?

- Varies, but typically no single source dominates.



Uncertainties in winter precipitation changes for the 2080s relative to 1961-90, at a 25km box in SE England

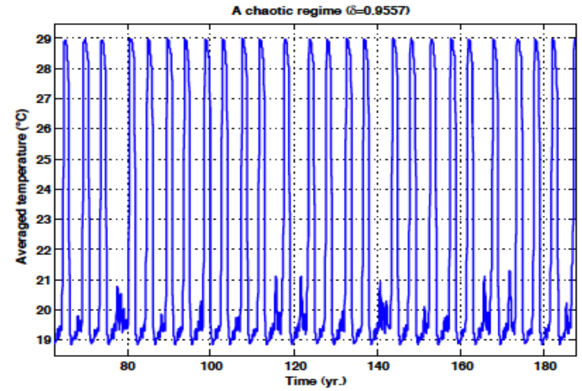
Source: Met Office



Courtesy Tim Palmer, 2009

Parameter dependence – I

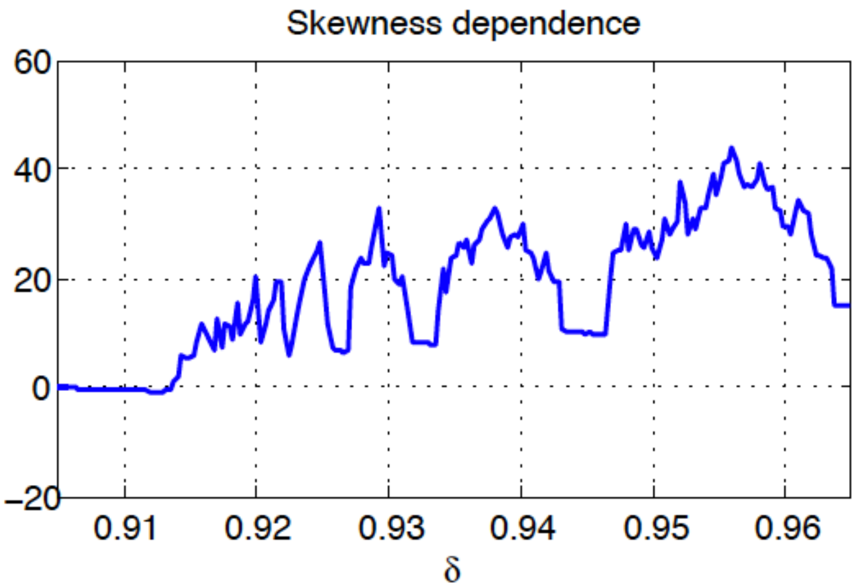
$\delta = 0.9557$

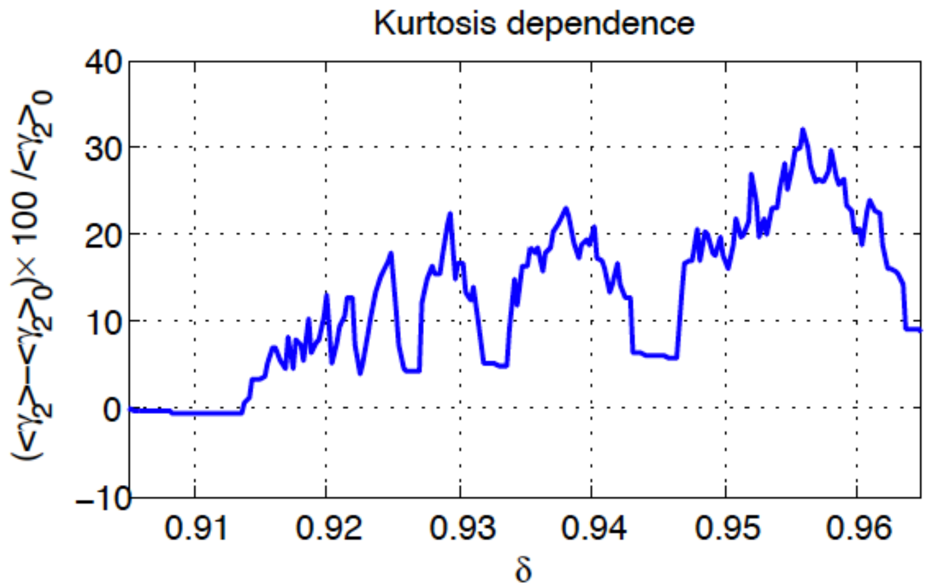


It can be smooth or it can be rough:
Niño-3 SSTs from intermediate coupled model for ENSO (Jin, Neelin & Ghil, 1994, 1996)

Skewness & kurtosis of the SSTs:
time series of 4000 years,

$$\Delta\delta = 3 \cdot 10^{-4}$$





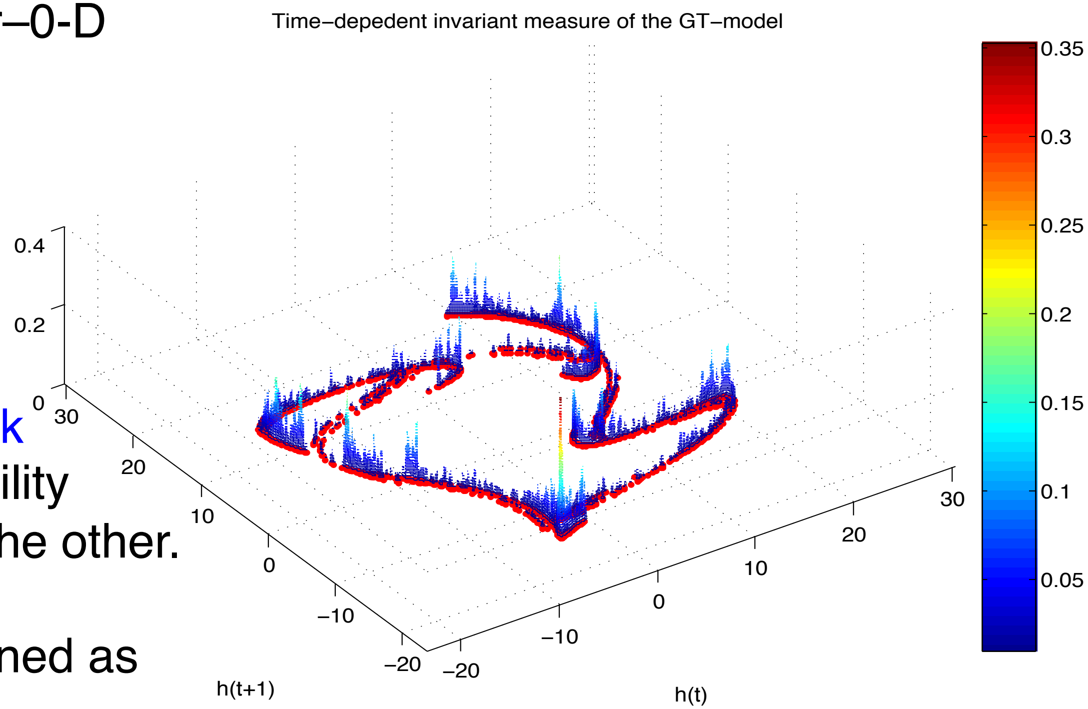
M. Chekroun (work in progress)

Pullback attractor and invariant measure of the GT model

The time-dependent pullback attractor of the GT model supports an invariant measure $\nu = \nu(t)$, whose density is plotted in 3-D perspective.

The plot is in delay coordinates *h*(*t*+1) *vs*. *h*(*t*) and the density is highly concentrated along 1-D filaments and, furthermore, exhibits sharp, near–0-D peaks on these filaments.

The Wasserstein distance d_W between one such configuration, at given parameter values, and another one, at a different set of values, is proportional to the work needed to move the total probability mass from one configuration to the other.



Climate sensitivity γ can be defined as

$$\gamma = \partial d_{\mathrm{W}} / \partial \tau$$

Exponential divergence vs. "coarse graining"

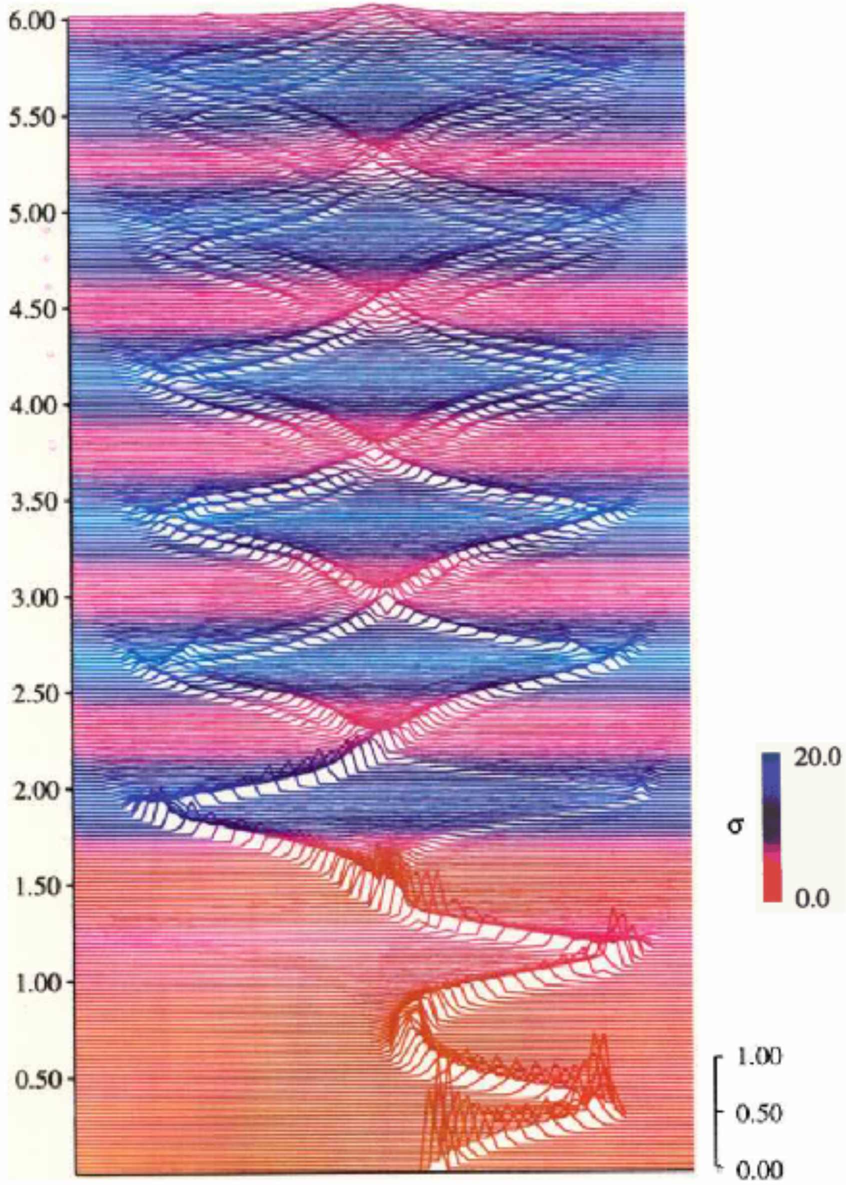
The classical view of dynamical systems theory is:

positive Lyapunov exponent ➔ trajectories diverge exponentially

But the presence of multiple regimes implies a much more structured behavior in phase space

Still, the probability distribution function (pdf), when calculated forward in time, is pretty smeared out

L. A. Smith (*Encycl. Atmos. Sci.*, 2003)



Global warming and "global weirding"

"CLIMATE STRANGE
FORGET GLOBAL WARMING—AND
GET READY for GLOBAL WEIRDING
BY BRYAN WALSH"

TIME MAGAZINE, Dec. 29, 2014 – Jan. 5, 2015

"***The New Rule***: For the next few (?) years, global warming will lead to colder, more brutal winters."

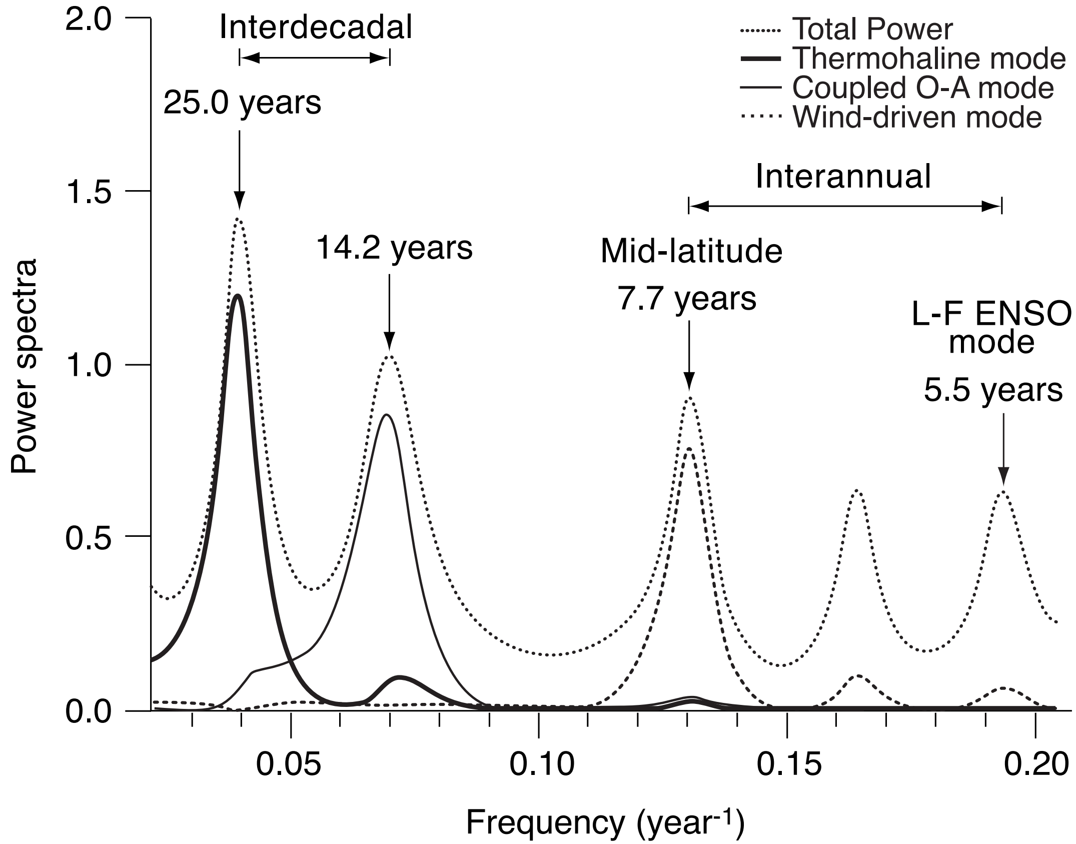
- Oh, thank you for the latest prediction from a science journalist — based on interesting but still rather tentative, & hotly debated, suggestions from a few media-loving (& vice-versa) researchers.

- And if this is so certain, why wasn't it predicted by IPCC(*) and other models BEFORE it happened?

(*) Intergovernmental Panel on Climate Change

SSA (prefilter) + (low-order) MEM

- "Stack" spectrum



In good agreement with MTM peaks of **Ghil & Vautard (1991, *Nature*)** for the Jones *et al.* (1986) temperatures & stack spectra of Vautard *et al.* (1992, *Physica D*) for the IPCC "consensus" record (both global), to wit 26.3, 14.5, 9.6, 7.5 and 5.2 years.

Peaks at 27 & 14 years also in Koch sea-ice index off Iceland (Stocker & Mysak, 1992), etc.
Plaut, Ghil & Vautard (1995, *Science*)

Concluding remarks, I – RDS and RAs

Summary

- A change of paradigm from **closed, autonomous systems** to **open, non-autonomous ones.**
- Random attractors are (i) spectacular, (ii) useful, and (iii) just starting to be explored for climate applications.

Work in progress

- Study the effect of specific stochastic parametrizations on model robustness.
- Applications to intermediate models and GCMs.
- Implications for climate sensitivity.
- Implications for predictability?

Concluding remarks, II –
Climate change & climate sensitivity

What do we know?

- It's getting warmer.
- We do contribute to it.
- So we should act as best we know and can!

What do we know less well?

- By how much?
  - Is it getting warmer …
  - Do we contribute to it …
- How does the climate system (atmosphere, ocean, ice, etc.) really work?
- How does natural variability interact with anthropogenic forcing?

What to do?

- Better understand the system and its forcings.
- Explore the models', and the system's, robustness and sensitivity
  - stochastic structural and statistical stability!
  - linear response = response function + susceptibility function!!