A Mathematical Theory of Climate Sensitivity:
A Tale of Deterministic & Stochastic Dynamical Systems

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Ecole Normale Supérieure, Paris, and
University of California, Los Angeles

Based on joint work with A. Bracco, M.D. Chekroun, D. Kondrashov,
H. Liu, J.C. McWilliams, J.D. Neelin, S. Pierini,
E. Simonnet, S. Wang & I. Zaliapin

Please visit these sites for more info.
http://www.atmos.ucla.edu/tcd/
http://www.environnement.ens.fr/
Overall Outline

• A selection of Peter Lax contributions

• Dynamical systems & climate (MG & friends)
P. D. Lax Contributions – Selected!

• The Lax equivalence theorem in numerical analysis
• Specific numerical schemes for hyperbolic PDEs
  – Lax-Friedrichs – first-order accurate, positive
  – Lax-Wendroff – second-order accurate
• Hyperbolic systems of conservation laws
  – the entropy condition
• $\infty$-dimensional Hamiltonian systems
  – Lax equation and Lax pairs for the KdV equation
  – Generalization to a large class of such PDEs
  – Extensions to localized coherent structures in geophysical flows
• “PDE Lax”
  – Lax-Milgram theorem & its application to the finite-element method
  – Hopf-Lax-Oleinik formula for the Hamilton-Jacobi equation
• The “Lax school”
The Lax equivalence theorem in numerical analysis

**Theorem.** Given the consistency of a finite-difference scheme for an evolution problem (i.e., that it formally solves the hyperbolic or parabolic PDE at hand), convergence $\iff$ stability (Lax & Richtmyer, *CPAM*, 1956).

**Proof.** Stability $\Rightarrow$ convergence is pretty easy to prove, while convergence $\Rightarrow$ stability is not, and the latter requires a functional-analysis trick.

**Remark.** In practice, the useful observation is that, if it’s consistent and it doesn’t blow up in your face, it will eventually converge, as $\Delta t \to 0$.

This result has been called the **Fundamental Theorem of Numerical Analysis:** “[It] gave us work to do, precise results to prove, something to accomplish with our analysis and our lives.”(*)

(*) G. Strang, in his review of the book

*Peter Lax, Mathematician: An Illustrated Memoir*, 2015,

by Reuben Hersh, American Mathematical Society, Providence, RI;
see *SIAM News*, May 1, 2015.
P. D. Lax Contributions – I, Numerical Analysis

- Practical finite-difference schemes for hyperbolic PDEs
  - Lax-Friedrichs scheme
    \[ u_t + a u_x = 0, \quad u_i^n = u(x_i, t_n), \quad x_i = i \Delta x, \quad t_n = n \Delta t; \]
    \[ u_i^{(n+1)} = \frac{1}{2} (u_i^{n+1} + u_i^{n-1}) - a \frac{\Delta t}{2 \Delta x} (u_i^{n+1} - u_i^{n-1}) \]

  The LF scheme is FTCS – forward in time & centered in space. It preserves positivity, & thus monotonicity of shocks in the nonlinear case, but is not terribly accurate, i.e., its truncation error is only of first order in \( \Delta t \) and \( \Delta x \).

  - Lax-Wendroff scheme
    The LW scheme achieves second-order accuracy by also using \( u_i^n \) in the Taylor expansion of \( u_x \), but positivity is lost; this loss leads to “ringing” or the Gibbs phenomenon in the computation of shocks.

  The quandary of accuracy vs. monotonicity was solved in the Ph.D. thesis of Amiram Harten (RIP) with Peter, by introducing total-variation diminishing (TVD) methods. Later, Ami and Stanley Osher realized the applicability and usefulness of TVD methods in “edge detection” and hence image compression.
Not every big wave is a solitary wave nor a soliton: it must (a) be local; (b) preserve its shape and (c) “non-interact.”
P. D. Lax Contributions – II, ∞-D Dynamical Systems

- **The KdV equation** – J. Scott Russell (1834) – observation + experiment
  - J. Boussinesq (1877) + D.J. Korteweg & G. De Vries (1895): KdV equation
    \[ u_t + uu_x + u_{xxx} = 0 \]
  - The KdV equation balances nonlinearity \( uu_x \) and dispersion \( u_{xxx} \) of the waves.
  - P.D. Lax (CPAM, 1968) associated with every solution \( u = u(t) \) of \( u_t = K(u) \) a linear Sturm-Liouville operator \( L_u \) and the linear PDE
    \[ L_t = BL - LB. \] (L)
    Here \( B_t \) is a one-parameter family of unitary operators, like \( L_u(t) \), \( (B, L) \) is a **Lax pair** and (L) is the **Lax equation**.

- Many other ∞-dimensional Hamiltonian problems have these nice properties, i.e., an infinite number of invariants = \{the eigenvalues of \( L \}\):
  - nonlinear Schrödinger equation, sine-Gordon equation, Toda lattice.
- **Localized coherent structures** in geophysical fluid dynamics (GFD):
  - 2-D modons and their 3-D generalizations, thermons (“hetons”), etc.
# The Lax School - I

**Peter David Lax**

**Biography MathSciNet**

Ph.D. New York University 1949

Dissertation: *Nonlinear System of Hyperbolic Partial Differential Equations in Two Independent Variables*

Advisor: Kurt Otto Friedrichs

Students:

Click [here](#) to see the students listed in chronological order.

<table>
<thead>
<tr>
<th>Name</th>
<th>School</th>
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<td>Steven Alpern</td>
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<td>Melvyn Ciment</td>
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<td>Charles Goldstein</td>
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<td>Amiram Harman</td>
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<td>Reuben Hersh</td>
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<td>George Logemann</td>
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<td>James Moeller</td>
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<td>Lucien Neustadt</td>
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<td>Sebastian Noelle</td>
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<td>Yirogos Smyrlis</td>
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<td>Blair Swartz</td>
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<td>Yahan Yang</td>
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</tbody>
</table>

According to our current on-line database, Peter Lax has 55 students and 555 descendants.

We welcome any additional information.
The Lax School - II

Peter Lax has 55 (former) Ph.D. students and 555 descendants.

But he was labeled “the most versatile mathematician of his generation” by the Abel Prize selection committee.

And I would say that his school extends to wherever good mathematics is done and to whoever does it, ...

... especially if they are as kind and charming as Peter.
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(*) Please see
Peter D. Lax: A Life in Mathematics, by M. Ghil
Outline – Dynamical systems & climate sensitivity

• The IPCC process: results and uncertainties
• Natural climate variability as a source of uncertainties
  – sensitivity to initial data  →  error growth
  – sensitivity to model formulation  →  see below!
• Uncertainties and how to fix them
  – structural stability and other kinds of robustness
  – non-autonomous and random dynamical systems (NDDS & RDS)
• Two illustrative examples
  – the Lorenz convection model
  – an El Niño–Southern Oscillation (ENSO) model
• A mathematical definition of climate sensitivity
• Conclusions and references
  – natural variability and anthropogenic forcing: the “grand unification”
  – selected bibliography
Motivation

• The *climate system* is highly **nonlinear and quite complex**.
• The system’s **major components** — the atmosphere, oceans, ice sheets — **evolve** on many time and space scales.
• Its **predictive understanding** has to rely on the system’s physical, chemical and biological modeling, but also on the thorough mathematical analysis of the models thus obtained: *the forest vs. the trees*.
• The **hierarchical modeling** approach allows one to give proper weight to the understanding provided by the models vs. their realism: back-and-forth between “toy” (conceptual) and **detailed** (“realistic”) **models**, and between **models** and **data**.
• How do we disentangle **natural variability** from the **anthropogenic forcing**: can we & should we, or not?
F. Bretherton's "horrendogram" of Earth System Science
Composite spectrum of climate variability

Standard treatment of frequency bands:

1. High frequencies – white noise (or “colored”)
2. Low frequencies – slow evolution of parameters


* “No known source of deterministic internal variability”

** 27 years – Brier (1968, *Rev. Geophys.*)
Climate and Its Sensitivity

Let’s say CO₂ doubles:
How will “climate” change?

1. Climate is in **stable equilibrium** (fixed point); if so, mean temperature will just shift gradually to its new equilibrium value.

2. Climate is **purely periodic**; if so, mean temperature will (maybe) shift gradually to its new equilibrium value. But how will the **period, amplitude and phase** of the limit cycle change?

3. And how about some “real stuff” now: chaotic + random?

Atmospheric CO₂ at Mauna Loa Observatory

Scripps Institution of Oceanography
NOAA Earth System Research Laboratory
Temperatures and GHGs

Greenhouse gases (GHGs) go up, temperatures go up:

It’s gotta do with us, at least a bit, doesn’t it?

Wikicommons, from Hansen et al. (PNAS, 2006); see also http://data.giss.nasa.gov/gistemp/graphs/
Global warming and its socio-economic impacts

Temperatures rise:
• What about impacts?
• How to adapt?

The answer, my friend, is blowing in the wind, i.e., it depends on the accuracy and reliability of the forecast …

Source: IPCC (2007), AR4, WGI, SPM
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### So what’s it gonna be like, by 2100?

<table>
<thead>
<tr>
<th>Phenomenon and direction of trend</th>
<th>Likelihood that trend occurred in late 20th century (typically post 1960)</th>
<th>Likelihood of a human contribution to observed trend[^b]</th>
<th>Likelihood of future trends based on projections for 21st century using SRES scenarios</th>
</tr>
</thead>
<tbody>
<tr>
<td>Warmer and fewer cold days and nights over most land areas</td>
<td>Very likely[^c]</td>
<td>Likely[^d]</td>
<td>Virtually certain[^d]</td>
</tr>
<tr>
<td>Warmer and more frequent hot days and nights over most land areas</td>
<td>Very likely[^c]</td>
<td>Likely (nights[^d])</td>
<td>Virtually certain[^d]</td>
</tr>
<tr>
<td>Warm spells/heat waves. Frequency increases over most land areas</td>
<td>Likely</td>
<td>More likely than not[^f]</td>
<td>Very likely</td>
</tr>
<tr>
<td>Heavy precipitation events. Frequency (or proportion of total rainfall from heavy falls) increases over most areas</td>
<td>Likely</td>
<td>More likely than not[^f]</td>
<td>Very likely</td>
</tr>
<tr>
<td>Area affected by droughts increases</td>
<td>Likely in many regions since 1970s</td>
<td>More likely than not[^f]</td>
<td>Likely</td>
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<tr>
<td>Intense tropical cyclone activity increases</td>
<td>Likely in some regions since 1970</td>
<td>More likely than not[^f]</td>
<td>Likely</td>
</tr>
<tr>
<td>Increased incidence of extreme high sea level (excludes tsunami[^f])</td>
<td>Likely</td>
<td>More likely than not[^f]</td>
<td>Likely</td>
</tr>
</tbody>
</table>

[^a]: Phenomenon of interest in the context of climate change.  
[^b]: Likelihood assessment based on scientific evidence and models.  
[^c]: Very likely indicates a high probability.  
[^d]: Likely indicates a moderate probability.  
[^e]: More likely than not indicates a slight probability.  
[^f]: Likely in many regions or some regions since 1970.
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The uncertainties might be *intrinsic*, rather than mere “tuning problems”

If so, maybe *stochastic structural stability* could help!

Might fit in nicely with recent taste for “stochastic parameterizations”

*The DDS dream of structural stability* (from Abraham & Marsden, 1978)
Consider the scalar, linear ordinary differential equation (ODE)
\[ \dot{x} = -\alpha x + \sigma t, \quad \alpha > 0, \quad \sigma > 0. \]
The autonomous part of this ODE, \( \dot{x} = -\alpha x \), is dissipative and all solutions \( x(t; x_0) = x(t; x(0) = x_0) \) converge to 0 as \( t \to +\infty \).

What about the non-autonomous, forced ODE? As the energy being put into the system by the forcing is dissipated, we expect things to change in time. In fact, if we “pull back” far enough, replace \( x(t; x_0) \) by \( x(s, t; x_0) = x(s, t; x(s) = x_0) \),

and let \( s \to -\infty \), we get the pullback attractor \( a = a(t) \) in the figure,
\[ a(t) = \frac{\sigma}{\alpha} (t - \frac{1}{\alpha}). \]
This theory is the counterpart for randomly forced dynamical systems (RDS) of the *geometric theory* of ordinary differential equations (ODEs). It allows one to treat stochastic differential equations (SDEs) — and more general systems driven by noise — as flows in (phase space) $\times$ (probability space).

$\text{SDE} \sim \text{ODE}, \ \text{RDS} \sim \text{DDS}, \ \text{L. Arnold} \ (1998) \sim \text{V.I. Arnol’d} \ (1983)$.

### Setting:

(i) A phase space $X$. **Example**: $\mathbb{R}^n$.

(ii) A probability space $(\Omega, \mathcal{F}, \mathbb{P})$. **Example**: The Wiener space $\Omega = C_0(\mathbb{R}; \mathbb{R}^n)$ with Wiener measure $\mathbb{P}$.

(iii) A model of the noise $\theta(t) : \Omega \to \Omega$ that preserves the measure $\mathbb{P}$, i.e. $\theta(t)\mathbb{P} = \mathbb{P}$; $\theta$ is called the driving system. **Example**: $W(t, \theta(s)\omega) = W(t + s, \omega) - W(s, \omega)$; it starts the noise at $s$ instead of $t = 0$.

(iv) A mapping $\varphi : \mathbb{R} \times \Omega \times X \to X$ with the cocycle property. **Example**: The solution operator of an SDE.
\( \varphi \) is a random dynamical system (RDS)

\( \Theta(t)(x, \omega) = (\theta(t)\omega, \varphi(t, \omega)x) \) is a flow on the bundle
A random attractor $A(\omega)$ is both \textit{invariant} and “pullback” \textit{attracting}:

(a) \textbf{Invariant}: $\varphi(t, \omega)A(\omega) = A(\theta(t)\omega)$.

(b) \textbf{Attracting}: $\forall B \subset X, \lim_{t \to \infty} \text{dist}(\varphi(t, \theta(-t)\omega)B, A(\omega)) = 0$ a.s.
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A snapshot of the RA, $\mathcal{A}(\omega)$, computed at a fixed time $t$ and for the same realization $\omega$; it is made up of points transported by the stochastic flow, from the remote past $t - T, T >> 1$.

We use multiplicative noise in the deterministic Lorenz model, with the classical parameter values $b = 8/3, \sigma = 10$, and $r = 28$.

Even computed pathwise, this object supports meaningful statistics.
We compute the probability measure on the R.A. at some fixed time $t$, and for a fixed realization $\omega$. We show a “projection”, $\int \mu(\omega)(x, y, z)dy$, with multiplicative noise: $dx_i=\text{Lorenz}(x_1, x_2, x_3)dt + \alpha x_i dW_t; i \in \{1, 2, 3\}$. 10 million of initial points have been used for this picture!
Still 1 Billion I.D., and $\alpha = 0.5$. Another one?
Sample measures evolve with time.

- Recall that these sample measures are the \textit{frozen statistics} at a time $t$ for a realization $\omega$.

- How do these \textit{frozen statistics} evolve with time?

- \textbf{Action!}
A day in the life of the Lorenz (1963) model’s random attractor, or LORA for short; see SI in Chekroun, Simonnet & Ghil (2011, Physica D)
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Physically closed system, modeled mathematically as autonomous system: neither deterministic (anthropogenic) nor random (natural) forcing.

The attractor is strange, but still fixed in time ~ “irrational” number.

Climate sensitivity ~ change in the average value (first moment) of the coordinates \((x, y, z)\) as a parameter \(\lambda\) changes.
Physically open system, modeled mathematically as non-autonomous system: allows for deterministic (anthropogenic) as well as random (natural) forcing.

The attractor is “pullback” and evolves in time ~ “imaginary” or “complex” number.

Climate sensitivity ~ change in the statistical properties (first and higher-order moments) of the attractor as one or more parameters ($\lambda, \mu, \ldots$) change.

Ghil (Encyclopedia of Atmospheric Sciences, 2nd ed., 2012)
Parameter dependence – I

It can be smooth or it can be rough: Niño-3 SSTs from intermediate coupled model for ENSO (Jin, Neelin & Ghil, 1994, 1996)

Skewness & kurtosis of the SSTs: time series of 4000 years,

$$\Delta \delta = 3 \cdot 10^{-4}$$

M. Chekroun (work in progress)
Sample measures for an NDDE model of ENSO

The Galanti-Tziperman (GT) model (JAS, 1999)

\[
\frac{dT}{dt} = -\epsilon_T T(t) - M_0(T(t) - T_{\text{sub}}(h(t))),
\]

\[
h(t) = M_1 e^{-\epsilon_m(\tau_1+\tau_2)} h(t - \tau_1 - \tau_2)
- M_2 \tau_1 e^{-\epsilon_m(\tau_1/2+\tau_2)} \mu(t - \tau_2 - \tau_1/2) T(t - \tau_2 - \tau_1/2)
+ M_3 \tau_2 e^{-\epsilon_m \tau_2/2} \mu(t - \tau_2/2) T(t - \tau_2/2).
\]

Neutral delay-differential equation (NDDE), derived from Cane-Zebiak and Jin-Neelin models for ENSO: \(T\) is East-basin SST and \(h\) is thermocline depth.

Seasonal forcing given by \(\mu(t) = 1 + \epsilon \cos(\omega t + \phi)\).

The pullback attractor and its invariant measures were computed.

Figure shows the changes in the mean, 2\(^{nd}\) & 4\(^{th}\) moment of \(h(t)\), along with the Wasserstein distance \(d_W\), for changes of 0–5\% in the delay parameter \(\tau_{K,0}\).

Note intervals of both smooth & rough dependence!
Pullback attractor and invariant measure of the GT model

The time-dependent pullback attractor of the GT model supports an invariant measure $\nu = \nu(t)$, whose density is plotted in 3-D perspective.

The plot is in delay coordinates $h(t+1)$ vs. $h(t)$ and the density is highly concentrated along 1-D filaments and, furthermore, exhibits sharp, near–0-D peaks on these filaments.

The Wasserstein distance $d_W$ between one such configuration, at given parameter values, and another one, at a different set of values, is proportional to the work needed to move the total probability mass from one configuration to the other.

Climate sensitivity $\gamma$ can be defined as $\gamma = \partial d_W / \partial \tau$
How to define climate sensitivity or, What happens when there’s natural variability?

This definition allows us to watch how “the earth moves,” as it is affected by both natural and anthropogenic forcing, in the presence of natural variability, which includes both chaotic & random behavior:

\[ \gamma = \frac{\partial dW}{\partial \tau} \]
Outline

• The IPCC process: results and uncertainties
• Natural climate variability as a source of uncertainties
  – sensitivity to initial data ➔ error growth
  – sensitivity to model formulation ➔ see below!
• Uncertainties and how to fix them
  – structural stability and other kinds of robustness
  – non-autonomous and random dynamical systems (NDDS & RDS)
• Two illustrative examples
  – the Lorenz convection model
  – an El Niño–Southern Oscillation (ENSO) model
• Linear response theory and climate sensitivity
• Conclusions and references
  – natural variability and anthropogenic forcing: the “grand unification”
  – selected bibliography
Concluding remarks, I – RDS and RAs

Summary

• A change of paradigm from closed, autonomous systems to open, non-autonomous ones.
• Random attractors are (i) spectacular, (ii) useful, and (iii) just starting to be explored for climate applications.

Work in progress

• Study the effect of specific stochastic parametrizations on model robustness.
• Applications to intermediate models and GCMs.
• Implications for climate sensitivity.
• Implications for predictability?
Lorenz (JAS, 1963)
Climate is deterministic and autonomous, but highly nonlinear.
Trajectories diverge exponentially, forward asymptotic PDF is multimodal.

Hasselmann (Tellus, 1976)
Climate is stochastic and noise-driven, but quite linear.
Trajectories decay back to the mean, forward asymptotic PDF is unimodal.

Grand unification (?)
Climate is deterministic + stochastic, as well as highly nonlinear.
Internal variability and forcing interact strongly, change and sensitivity refer to both mean and higher moments.
Concluding remarks, II – Climate change & climate sensitivity

What do we know?
• It’s getting warmer.
• We do contribute to it.
• So we should act as best we know and can!

What do we know less well?
• By how much?
  – Is it getting warmer …
  – Do we contribute to it …
• How does the climate system (atmosphere, ocean, ice, etc.) really work?
• How does natural variability interact with anthropogenic forcing?

What to do?
• Better understand the system and its forcings.
• Explore the models’, and the system’s, robustness and sensitivity
  – stochastic structural and statistical stability!
  – linear response = response function + susceptibility function!!
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• Explore the models’, and the system’s, robustness and sensitivity
  – stochastic structural and statistical stability!
  – linear response = response function + susceptibility function!!
Some general references

Reserve slides
Unfortunately, things aren’t all that easy!

What to do?

Try to achieve better interpretation of, and agreement between, models …


Natural variability introduces additional complexity into the anthropogenic climate change problem

The most common interpretation of observations and GCM simulations of climate change is still in terms of a scalar, linear Ordinary Differential Equation (ODE)

\[
c \frac{dT}{dt} = -kT + Q = \sum k_i \text{ feedbacks (+ve and -ve)} \\
Q = \sum Q_j \text{ sources & sinks} \\
Q_j = Q_j(t)
\]

Linear GHG effect

Linear response to CO₂ vs. observed change in T

Hence, we need to consider instead a system of nonlinear Partial Differential Equations (PDEs), with parameters and multiplicative, as well as additive forcing (deterministic + stochastic)

\[
\frac{dX}{dt} = N(X, t, \mu, \beta)
\]
Global warming and its socio-economic impacts—II

Temperatures rise:
• What about impacts?
• How to adapt?

AR5 vs. AR4
A certain air of *déjà vu*:
GHG “scenarios” have been replaced by “representative concentration pathways” (RCPs), more dire predictions, but the uncertainties remain.

Source: IPCC (2013), AR5, WGI, SPM
Interdecadal oscillations and the warming trend in global temperature time series

M. Ghil & R. Vautard

THE ability to distinguish a warming trend from natural variability is critical for an understanding of the climatic response to increasing greenhouse-gas concentrations. Here we use singular spectrum analysis\(^1\) to analyse the time series of global surface air temperatures for the past 135 years\(^2\), allowing a secular warming trend and a small number of oscillatory modes to be separated from the noise. The trend is flat until 1910, with an increase of 0.4 °C since then. The oscillations exhibit interdecadal periods of 21 and 16 years, and interannual periods of 6 and 5 years. The interannual oscillations are probably related to global aspects of the El Niño-Southern Oscillation (ENSO) phenomenon\(^3\). The interdecadal oscillations could be associated with changes in the extratropical ocean circulation\(^4\). The oscillatory components have combined (peak-to-peak) amplitudes of 0.2 °C, and therefore limit our ability to predict whether the inferred secular warming trend of 0.005 °Cyr\(^{-1}\) will continue. This could postpone incontrovertible detection of the greenhouse warming signal for one or two decades.
Multiple scales of motion: Space-time organization

- The most active scales lie along a diagonal in this space vs. time plot.
- Why this is so is far from clear as of now.
- We’ll deal with weather first, then climate.

N.B. A high-variability ridge lies close to the diagonal of the plot (cf. also Fraedrich & Böttger, 1978, JAS)

* $\text{LFV} \approx 10\text{–}100$ days (intraseasonal)
How important are different sources of uncertainty?

- Varies, but typically no single source dominates.

Source: Met Office
Ensemble forecast of Lothar (surface pressure)
Start date 24 December 1999: Forecast time T+42 hours

Courtesy Tim Palmer, 2009
The classical view of dynamical systems theory is:
positive Lyapunov exponent \(\Rightarrow\) trajectories diverge exponentially

But the presence of multiple regimes implies a much more structured behavior in phase space

Still, the probability distribution function (pdf), when calculated forward in time, is pretty smeared out

Global warming and “global weirding”

“CLIMATE STRANGE
FORGET GLOBAL WARMING—AND GET READY for GLOBAL WEIRDING
BY BRYAN WALSH”


“The New Rule: For the next few (?) years, global warming will lead to colder, more brutal winters.”

- Oh, thank you for the latest prediction from a science journalist — based on interesting but still rather tentative, & hotly debated, suggestions from a few media-loving (& vice-versa) researchers.

- And if this is so certain, why wasn’t it predicted by IPCC(*) and other models BEFORE it happened?

(*) Intergovernmental Panel on Climate Change
Transitions Between Blocked and Zonal Flows in a Barotropic Rotating Annulus with Topography

**Zonal Flow**
13–22 Dec. 1978

**Blocked Flow**
10–19 Jan. 1963

Fig. 1. Atmospheric pictures of (A) zonal and (B) blocked flow, showing contour plots of the height (m) of the 700-hPa (700 mbar) surface, with a contour interval of 60 m for both panels. The plots were obtained by averaging 10 days of twice-daily data for (A) 13 to 22 December 1978 and (B) 10 to 19 January 1963; the data are from the National Oceanic and Atmospheric Administration’s Climate Analysis Center. The nearly zonal flow of (A) includes quasi-stationary, small-amplitude waves (32). Blocked flow advects cold Arctic air southward over eastern North America or Europe, while decreasing precipitation in the continent’s western part (26).

Weeks, Tian, Urbach, Ide, Swinney, & Ghil (*Science*, 1997)
In good agreement with MTM peaks of Ghil & Vautard (1991, Nature) for the Jones et al. (1986) temperatures & stack spectra of Vautard et al. (1992, Physica D) for the IPCC “consensus” record (both global), to wit 26.3, 14.5, 9.6, 7.5 and 5.2 years.

Peaks at 27 & 14 years also in Koch sea-ice index off Iceland (Stocker & Mysak, 1992), etc. Plaut, Ghil & Vautard (1995, Science)
Modeled Climate Sensitivity

Climate sensitivity as estimated from a series of “snapshot” simulations of paleoclimate using HadCM3.

Courtesy of Paul J. Valdes
**Parameter dependence – II**

When it is smooth, one can optimize a GCM’s single-parameter dependence

ICTP AGCM (Neelin, Bracco, Luo, McWilliams & Meyerson, *PNAS*, 2010)
Parameter dependence – III

Multi-objective algorithms avoid arbitrary weighting of criteria in a unique cost function:

Optimization algorithms that are $\mathcal{O}(N)$ and $\mathcal{O}(N^2)$, rather than $\mathcal{O}(S^N)$, where $N$ is the number of parameters and $S$ is the sampling density.

ICTP AGCM (Neelin, Bracco, Luo, McWilliams & Meyerson, *PNAS*, 2010)
GHGs rise!

It’s gotta do with us, at least a bit, ain’t it?
But just how much?

IPCC (2007)
AR4 adjustment of 20th century simulation

Hindcasts and Forecasts of Global Mean Temperature

Ed Tredger (PhD thesis, LSE, 2009)

L.A. (“Lenny”) Smith (2009)
personal communication
We’ve just shown that:

$$|x(t, s; x_0) - a(t)| \xrightarrow{s \to -\infty} 0 \; ; \text{for every } t \text{ fixed},$$

and for all initial data $x_0$, with $a(t) = \frac{\sigma}{\alpha} (t - 1/\alpha)$.

We’ve just encountered the concept of pullback attraction; here $\{a(t)\}$ is the pullback attractor of the system (1).

What does it mean physically?

The pullback attractor provides a way to assess an asymptotic regime at time $t$ — the time at which we observe the system — for a system starting to evolve from the remote past $s$, $s \ll t$.

This asymptotic regime evolves with time: it is a dynamical object.

Dissipation now leads to a dynamic object rather than to a static one, like the strange attractor of an autonomous system.
We’ve just shown that:

\[ |x(t, s; x_0) - a(t)| \xrightarrow{s \to -\infty} 0 \text{ ; for every } t \text{ fixed}, \]

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Dissipation now leads to a dynamic object rather than to a static one, like the strange attractor of an autonomous system.
This theory is the counterpart for randomly forced dynamical systems (RDS) of the geometric theory of ordinary differential equations (ODEs). It allows one to treat stochastic differential equations (SDEs) — and more general systems driven by noise — as flows in (phase space) \( \times \) (probability space).

**Setting:**

(i) A phase space \( X \). **Example:** \( \mathbb{R}^n \).

(ii) A probability space \( (\Omega, \mathcal{F}, P) \). **Example:** The Wiener space \( \Omega = C_0(\mathbb{R}; \mathbb{R}^n) \) with Wiener measure \( P \).

(iii) A model of the noise \( \theta(t) : \Omega \to \Omega \) that preserves the measure \( P \), i.e. \( \theta(t)P = P \); \( \theta \) is called the driving system. **Example:** \( W(t, \theta(s)\omega) = W(t + s, \omega) - W(s, \omega) \); it starts the noise at \( s \) instead of \( t = 0 \).

(iv) A mapping \( \varphi : \mathbb{R} \times \Omega \times X \to X \) with the cocycle property. **Example:** The solution operator of an SDE.
Timmerman & Jin (Geophys. Res. Lett., 2002) have derived the following low-order, tropical-atmosphere–ocean model. The model has three variables: thermocline depth anomaly $h$, and SSTs $T_1$ and $T_2$ in the western and eastern basin.

\[
\begin{align*}
\dot{T}_1 &= -\alpha(T_1 - T_r) - \frac{2\varepsilon u}{L}(T_2 - T_1), \\
\dot{T}_2 &= -\alpha(T_2 - T_r) - \frac{w}{H_m}(T_2 - T_{sub}), \\
\dot{h} &= r(-h - bL\tau/2).
\end{align*}
\]

The related diagnostic equations are:

\[
\begin{align*}
T_{sub} &= T_r - \frac{T_r - T_{r0}}{2}\left[1 - \tanh(H + h_2 - z_0)/h^*\right] \\
\tau &= \frac{a}{\beta}(T_1 - T_2)[\xi_t - 1].
\end{align*}
\]

- $\tau$: the wind stress anomalies, $w = -\beta\tau/H_m$: the equatorial upwelling.
- $u = \beta L\tau/2$: the zonal advection, $T_{sub}$: the subsurface temperature.

Wind stress bursts are modeled as white noise $\xi_t$ of variance $\sigma$, and $\varepsilon$ measures the strength of the zonal advection.
Random attractors: the frozen statistics

Random Shil’nikov horseshoes

Horseshoes can be noise-excited,
left: a weakly-perturbed limit cycle, right: the same with larger noise.

Golden: most frequently-visited areas; white 'plus' sign: most visited.

$\sigma=0.005$  $\sigma=0.05$
An episode in the random’s attractor life

Michael Ghil

Climate Change and Climate Sensitivity
Devil's Bleachers in a 1-D ENSO Model

Ratio of ENSO frequency to annual cycle

But deterministic chaos doesn’t explain all: there are many other sources of irregularity!

- The energy spectrum of the atmosphere and ocean is “full”: all space & time scales are active and they all contribute to forecasting uncertainties.
- Still, one can imagine that the longest & slowest scales contribute most to the longest-term forecasts.
- “One person’s signal is another person’s noise.”

After Nastrom & Gage (JAS, 1985)
Climatic uncertainties & moral dilemmas

♥ … keep today’s climate for tomorrow?

♥ Feed the world today or…