Scalings for eddy buoyancy fluxes across prograde shelf/slope fronts

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Key Points:

• Process simulations of eddy buoyancy fluxes across prograde shelf and slope fronts are conducted.
• The GEOMETRIC scaling and the Eady scale-based scaling for eddy buoyancy fluxes are adapted to continental slopes.
• A basis for parameterizing eddy buoyancy fluxes across prograde fronts is presented.

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Abstract
Depth-averaged eddy buoyancy diffusivities across continental shelves and slopes are investigated using a suite of eddy-resolving, process-oriented simulations of prograde frontal currents characterized by isopycnals tilted in the opposite direction to the seafloor, a flow regime commonly found along continental margins under downwelling-favorable winds or occupied by buoyant boundary currents. The diagnosed cross-slope eddy diffusivity varies by up to three orders of magnitude, decaying from $O(10^4 \text{ m}^2/\text{s})$ in the relatively flat-bottomed region to $O(10 \text{ m}^2/\text{s})$ over the steep continental slope, consistent with previously reported suppression effects of steep topography on baroclinic eddy fluxes. To theoretically constrain the simulated cross-slope eddy fluxes, we examine extant scalings for eddy buoyancy diffusivities across prograde shelf/slope fronts and in flat-bottomed oceans. Among all tested scalings, the GEOMETRIC framework developed by Marshall et al. (2012, https://doi.org/10.1175/JPO-D-11-048.1) and a parametrically-similar Eady scale-based scaling proposed by Jansen et al. (2015, https://doi.org/10.1016/j.ocemod.2015.05.007) most accurately reproduce the diagnosed eddy diffusivities across the entire shelf-to-open-ocean analysis regions in our simulations. This result relies upon the incorporation of the topographic suppression effects on eddy fluxes, quantified via analytical functions of the slope Burger number, into the scaling prefactor coefficients. The predictive skills of the GEOMETRIC and Eady scale-based scalings are shown to be insensitive to the presence of along-slope topographic corrugations. This work lays a foundation for parameterizing eddy buoyancy fluxes across large-scale prograde shelf/slope fronts in coarse-resolution ocean models.

Plain Language Summary
Understanding the future climate relies on numerical predictions from climate models. However, these models are limited in accuracy because they cannot resolve all crucial processes in the climate system due to computational resource limitations. One such process is the oceanic “mesoscale” turbulence (swirling ocean flows that are tens to hundreds of kilometers wide) across continental margins. These flows impact coastal circulation and ecosystems by mediating material exchanges between coastal and open-ocean environments. By running computer simulations that can explicitly resolve mesoscale turbulence, we show that heat transport by mesoscale flows becomes less efficient over continental margins than that in the open ocean, due to the presence of the sloping seafloor. After taking this reduced efficiency into account, we are able to predict the heat transport by mesoscale flows across continental margins by adapting established theories for the open-ocean environment. This work provides a basis for improving the performance of climate models, especially near coastal regions.

1 Introduction
Continental shelves and slopes host a variety of baroclinic ocean currents, which can arise from tidal mixing fronts (Simpson & Hunter, 1974; Badin et al., 2009; Brink, 2012), shelf buoyant plumes (Hetland, 2017), intrusion of warm waters into convective basins (Spall, 2004, 2010a; Isachsen, 2011), shelf break fronts (Lozier & Reed, 2005; Zhang & Gawarkiewicz, 2015), and wind-driven upwelling and downwelling systems (Gan & Allen, 2002; Brink, 2016). These currents are commonly susceptible to baroclinic instabilities, from which a mesoscale eddy field develops and mediates the cross-slope exchanges (e.g., Cimoli et al., 2017). Prominent examples include eddies shed by the Norwegian Atlantic Current that are observed to carry warm and saline Atlantic water into the Lofoten Basin and precondition the wintertime deep convection (Spall, 2010b; Segtnan et al., 2011). Eddies spawned from the Antarctic Slope Front are found to drive the onshore heat fluxes, which control the melting of ice shelves in a warming climate (Stewart et al., 2018; Thompson et al., 2018). Along the Mid-Atlantic Bight, onshore eddy transfer of Atlantic water is observed to substantially alter the habitats and thus life cycles of commercially important
marine species (Gawarkiewicz et al., 2018). In the Eastern Boundary Upwelling Systems, both observational records and numerical models reveal the suppressive effect of mesoscale eddies on biological production via the offshore export of organic matters (Gruber et al., 2011).

Motivated by the crucial impact of mesoscale eddies on the cross-slope transfer, numerous studies have been devoted to the stability properties of ocean currents over continental shelves and slopes. Early works in this category built upon the modified quasi-geostrophic (QG) Eady (1949) or Phillips (1954) models, in which the slope parameter $\delta$, defined as the ratio between the topographic slope and the isopycnal slope, governs the baroclinic stability of an along-slope current (Blumsack & Gierasch, 1972; Mechos, 1980; Isachsen, 2011). Specifically, the QG theory predicts that if the isopycnals tilt in the opposite direction as the ocean bed ($\delta < 0$, referred to as “prograde”; Poulin et al., 2014) then a steepened topographic slope results in less vigorous baroclinic instabilities occurring at shorter wavelengths. By contrast, if the isopycnals and ocean bed tilt in the same direction ($\delta > 0$, referred to as “retrograde”), the exact opposite trend arises, but the current is completely stabilized for $\delta > 1$. These QG-based ideas were later extended to primitive equation systems or shallow-water models for investigating the topographic suppression on baroclinic instabilities over shelves and slopes (e.g., Spall, 2004; Isachsen, 2011; Pennel et al., 2012; C. Chen & Kamenkovich, 2013; Stewart & Thompson, 2013; Poulin et al., 2014; Cimoli et al., 2017; Ghaffari et al., 2018; S.-N. Chen et al., 2020). Other works suggest that it is the ratio between the internal deformation radius and the width of topographic slope, which can be approximated by the slope Burger number, determines the baroclinic stability and thus eddy characteristics of shelf and slope currents (e.g., Stern et al., 2015; Hetland, 2017). Specifically, a shelf or slope current associated with a greater slope Burger number is found to be more baroclinically stable (Brink, 2012; Brink & Cherian, 2013; Brink, 2016).

Apart from identifying the condition under which baroclinic eddies may develop over shelves and slopes, the aforementioned studies serve as a basis upon which the cross-slope transfer driven by these eddies can be quantified and thus parameterized in coarse resolution ocean climate models. Modern climate models remain incapable of resolving the ocean mesoscale over continental shelves and slopes owing to the significant decrease of internal deformation radius across the shoaling bathymetry (Hallberg, 2013). This issue is exacerbated toward regions at higher latitudes, where smaller deformation radii are found (Chelton et al., 1998; LaCasce & Groeskamp, 2020) and flows are largely steered by sloping topography (Isachsen et al., 2003; Isachsen, 2011, 2015). Recent modeling studies estimated a horizontal grid spacing of $1/72^o$ as the minimum requirement to resolve most (~70%) of mesoscale processes over continental shelves and slopes globally (Holt et al., 2017). Such a fine resolution contrasts with the fact that even state-of-the-art coupled global simulations are employing an ocean component with a grid spacing of $\sim 1/24^o$ (e.g., the GEOS-MITgcm simulation; Stroback et al., 2022) and that most IPCC-class Earth System Models still adopt a horizontal resolution of $1^o$ or coarser for the ocean component to accommodate lengthy integration and large ensembles.

The pressing need to parameterize mesoscale eddies over continental shelves and slopes motivated several authors to quantify the cross-slope eddy buoyancy diffusivity under the framework of Gent and McWilliams (1990), which, in addition to that of Redi (1982), has served as the standard scheme for parameterizing the open-ocean eddy-driven transports in today’s climate models (e.g., Gent, 2011). The Gent and McWilliams (1990) scheme specifically prescribes an eddy buoyancy diffusivity to link the adiabatic eddy buoyancy fluxes with the large-scale buoyancy gradients based on a horizontally down-gradient diffusive closure, ensuring that the eddy parameterization extracts available potential energy from the resolved flow.

In light of the modified Eady (1949) solutions (Blumsack & Gierasch, 1972), Stipa (2004) constructed an eddy buoyancy diffusivity following the scaling argument of Stone (1972) by utilizing the growth rate and wavelength of the most unstable baroclinic mode

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over the sloping bottom. Spall (2004) also formulated a scaling based on the modified Eady (1949) solutions for eddy heat fluxes across continental slopes in convective marginal seas. The approach of Stipa (2004) was later examined by Isachsen (2011) against eddy-resolving primitive equation simulations over steep slopes. This comparison reveals qualitative similarities between the Eady (1949) prediction and the simulated eddy buoyancy diffusivity for the case of $\delta < 0$.

A few empirical approaches have also been developed for parameterizing eddies across shelves and slopes. For instance, Stewart and Thompson (2013) employed a simple analytical fit for the cross-slope eddy buoyancy diffusivity as a function of $\delta$, based on an idealized simulation of the Antarctic Slope Front. This scaling was later adopted by Si et al. (2022) to construct a reduced-order model for the momentum balance of the Antarctic Slope Current. Brink (2012) and Brink and Cherian (2013) performed idealized eddy-resolving simulations of baroclinic instabilities fueled by tidal mixing fronts, across which the eddy buoyancy diffusivity was framed as the product of an empirically-derived analytical function of the slope Burger number and a reference flow-dependent diffusivity. Similar approaches were later adopted to quantify the eddy buoyancy diffusivity across slumping fronts over continental shelves forced by episodic upwelling- and downwelling-favorable winds (Brink, 2016).

In this work, we assess the relative merits of various scalings proposed by previous authors (Spall, 2004; Brink, 2012; Brink & Cherian, 2013; Brink, 2016) in quantifying the eddy buoyancy diffusivity across continental shelves and slopes by employing a suite of eddy-resolving, process-oriented simulations of mesoscale turbulence in along-slope currents characterized by $\delta < 0$ (i.e., prograde frontal currents). This parameter regime complements that studied by Wang and Stewart (2020), who specifically focused on quantifying eddy buoyancy fluxes for the case of $\delta > 0$ (i.e., retrograde frontal currents). Based on this assessment, we identify the key physical parameters controlling the eddy buoyancy fluxes and thus develop a numerically-implementable scaling for the cross-slope eddy buoyancy diffusivity.

The present work differs from and supplements the aforementioned studies in two major aspects:

(i) Calculations and scalings of the cross-slope eddy buoyancy diffusivity in previous works were confined to specific regions of the shelf and slope. For instance, Brink (2012) calculated the diffusivities at the surface and mid-depth across the center of a tidal mixing front, which were then averaged to obtain a bulk estimate. Isachsen (2011) calculated eddy buoyancy fluxes at locations centered over the continental slope in eddy-resolving simulations. Brink (2016) quantified the eddy buoyancy diffusivities at locations of maximum eddy energy in simulations of wind-forced continental shelf flows. For studies addressing initial value problems (e.g., by focusing on the “spin-down” flow phases; see Pennel et al., 2012; Brink, 2012; Brink & Cherian, 2013; Zhang & Gawarkiewicz, 2015; Brink, 2016; Cimoli et al., 2017; Hetland, 2017; S.-N. Chen et al., 2020), these calculations must also be temporally-confined. Here we focus on the dynamically equilibrated eddy fluxes across both sloping- and relatively flat-bottomed regions. Such a spatial transition is then quantified and incorporated into the scaling for the cross-slope eddy buoyancy diffusivity.

(ii) As with Wang and Stewart (2020), we inspect whether existing eddy parameterizations adopted by today’s ocean climate models, such as the GEOMETRIC scheme (an explicit linear scaling of the total eddy energy derived from a mathematical bound of the mesoscale eddy stress tensor; see D. P. Marshall et al., 2012; Maddison & Marshall, 2013; Bachman et al., 2017; Poulsen et al., 2019) and the eddy energy-based mixing length scheme (Eden & Greatbatch, 2008; Cessi, 2008; Jansen et al., 2015, 2019; Kong & Jansen, 2021), may be adapted to continental shelves and slopes. Answers to this question lead to a convenient “slope-aware” modification of a parameterization scheme.
that has already been tested for flat-bottomed ocean environments and implemented in ocean climate models.

The rest of this article is organized as follows. Section 2 describes the model configurations adopted in this study. In Section 3, we quantify the simulated eddy buoyancy diffusivity across prograde fronts, evaluate the skill of extant scalings of the eddy buoyancy diffusivity against our eddy-resolving simulations, and propose a pair of slope-aware scalings for constraining the depth-averaged eddy buoyancy diffusivity across our channel model. In Section 4, we discuss the implications of the parameter dependencies of various scalings tested in this study and assess the transferability of our proposed slope-aware scalings to shelves and slopes featuring topographic canyons and ridges. Our conclusions and an outlook on future work follow in Section 5.

2 Model Setup

In this section, we describe the model configuration of our eddy-resolving simulations. All experiments are conducted using the MIT general circulation model (MITgcm hereafter, J. Marshall et al., 1997), which has been adopted by various previous authors to investigate eddy processes across continental shelves and slopes (e.g., Spall, 2004; Stewart & Thompson, 2016; Wang & Stewart, 2018; Manucharyan & Isachsen, 2019; Yang et al., 2021). The range of most configuration parameters in this study follows that of Wang and Stewart (2020), but the focus here is specifically on prograde fronts (i.e., \( \delta < 0 \)).

2.1 Reference Model Configuration

Table 1 summarizes the values of physical parameters adopted in our reference model run. As in Wang and Stewart (2020), we consider a zonal channel (Figure 1a) with an idealized continental shelf of 500 m depth located at the southern boundary of the model domain. The shelf is deeper than most realistic continental shelves (e.g., Cacchione et al., 2002), which ensures that mesoscale processes are adequately resolved across the shelf and slope. The ocean deepens offshore across an idealized continental slope and reaches a maximum depth of 4000 m at the northern boundary. More precisely, the model bathymetry \( z = H(x, y) \) is defined by a hyperbolic tangent function,

\[
H(x, y) = -Z_s - \frac{1}{2} H_s \tanh \left( \frac{y - \tilde{Y}}{W_s} \right).
\]

\[
\tilde{Y}(x) = \begin{cases} 
Y_s & \text{Smooth continental slopes}, \\
Y_s - Y_t \sin \left( 2\pi M_t \frac{x}{L_x} \right) & \text{Corrugated continental slopes},
\end{cases}
\]

where \( x \in [-L_x/2, L_x/2] \) is the along-slope distance (longitude) from the domain center, \( y \in [0, L_y] \) is the offshore distance (latitude), \( Z_s = 2250 \) m denotes the slope mid-depth, \( H_s = 3500 \) m represents the shelf height, \( W_s = 50 \) km is the slope half-width, and \( \tilde{Y} \) indicates the mid-slope position, which can be varied longitudinally via the adjustment of the bathymetric mode number \( M_t \) and the onshore/offshore excursion amplitude \( Y_t \) around its mean latitude of \( y = Y_s = 200 \) km. Such bathymetric variations correspond to topographic canyons and ridges found ubiquitously along realistic continental margins (Harris & Whiteway, 2011). In our reference run, both \( M_t \) and \( Y_t \) are set identically to zero (i.e., \( \tilde{Y} = Y_s \), and thus the channel is zonally-symmetric). The channel spans 800 km and 500 km in the along-slope and cross-slope directions, respectively. Throughout this work, we will use “along-slope” and “longitudinal” or “zonal” interchangeably, and similarly for “cross-slope” with “latitudinal” or “meridional”. The channel is posed on an \( f \)-plane, with a constant Coriolis frequency \( f_0 = 1 \times 10^{-4} \) s\(^{-1} \) in the reference simulation, as changes in ocean depth dominate the
background potential vorticity (PV) gradient. Therefore, the continental shelf and slope
can be viewed as being oriented in any direction relative to the meridians.

We use a horizontal grid spacing of 1 km and 70 vertical levels, with vertical grid spacing
increasing from around 10 m at the surface to over 100 m at the ocean bed. Partial grid cells
with a minimum non-dimensional fraction of 0.1 are adopted to improve the representation
of flows near the sloping ocean bed (Adcroft et al., 1997; Pacanowski & Gnanadesikan,
1998). The finer horizontal grid spacing than that adopted by Wang and Stewart (2020) is
found to effectively suppress the intermittency of baroclinic instabilities found in prograde
fronts (Huneke et al., 2019), which may prevent the flow field from reaching a statistically
steady state (Stewart & Hogg, 2017).

The model is forced by a steady zonal wind stress with the profile defined by

$$
\tau = +\tau_0 \cdot \sin^2 \left( \frac{\pi y}{L} \right), \quad 0 < y < L.
$$

Here $\tau_0 = 0.05 \text{ N/m}^2$ denotes the peak strength of the wind stress, located at the mid-slope
position (i.e., $y = Y_s = 200$ km), $L = 400$ km represents the width of wind stress, and the
positive sign on the right hand side of (2) corresponds to the prograde (i.e., eastward or
downwelling-favorable, Figure 1a) direction of wind stress. A quadratic drag with coefficient
$C_d = 2.5 \times 10^{-3}$ is imposed at the ocean bed, which allows for momentum and energy
extractions.

Periodic boundary conditions are employed in the along-slope direction. No-normal-
flow conditions are imposed at the shoreward and offshore boundaries. At the northern
boundary, the potential temperature is restored toward a reference profile, which decays
exponentially from 10 $^\circ$C at the sea surface to 0 $^\circ$C at the seafloor, across a sponge layer of
50 km width with a minimum relaxation timescale of 14 days. This sponge layer maintains
the vertical stratification in the open ocean and effectively fixes the first baroclinic Rossby
deflection radius,

$$
L_d = \frac{1}{\pi f_0} \int_{-|H|}^{0} N \, dz,
$$

at approximately 18 km at the northern boundary, where $N$ denotes the buoyancy frequency
($L_d$ is roughly 4 km and 15 km at the southern boundary and the mid-slope position,
respectively, during the dynamic equilibrium in the reference run). In addition, a linear
cross-slope potential temperature profile that decays from 15 $^\circ$C at the southern boundary
to 10 $^\circ$C at $y = L_y - 50$ km offshore is prescribed in top 10 m following Haney (1971)
with a relaxation timescale of approximately 30 days. This surface temperature relaxation
maintains the horizontal stratification over the shelf and slope under the prograde wind
forcing. Additional tests reveal no qualitative sensitivity of the simulated flow to the range
of prescribed cross-slope surface temperature (not shown).

No parameterization of the surface mixed layer is included, though a large implicit ver-
tical diffusivity (100 m$^2$/s) is imposed to rectify instances of unstable vertical stratification.
A horizontal biharmonic viscosity with Courant-Friedrichs-Lewy number of 0.1 is employed
to maintain the numerical stability (e.g., Stewart & Thompson, 2015).

### 2.2 Experiments and Model Integration

A total of 18 simulations, as listed in Table 2, are performed. Each of these simulations
has at least one of its physical parameters (i.e., the peak wind strength $\tau_0$, the thermal
expansion coefficient $\alpha_\theta$, the Coriolis frequency $f_0$, the slope half-width $W_s$, the along-slope
bathymetric mode number $M_t$, and the bathymetric excursion amplitude $Y_t$) perturbed
relative to the reference model run.
Based on the bathymetric shape, experiments in Table 2 are divided into two categories: “Smooth” simulations (i.e., simulations with no along-slope bathymetric variation) and “Corrugated” simulations (i.e., simulations with finite along-slope bathymetric variations). For “Corrugated” simulations, the channel width is further expanded to 600 km to minimize the influence of the northern sponge layer on the simulated flow when the offshore excursions of the continental slope are large. In subsequent discussions, we first analyze the eddy buoyancy diffusivity across “Smooth” continental shelves and slopes, from which scalings for the eddy diffusivity are developed. These scalings are then tested against the “Corrugated” simulations to assess their potential utilities across much more complex continental margins in the real ocean (Harris & Whiteway, 2011).

All simulations integrate the three-dimensional, vector-invariant form of the hydrostatic, Boussinesq momentum equations coupled with a linearized equation of state that depends on potential temperature only. Each simulation was spun up from a resting state with a coarse 4-km resolution for 10 years, after which the solutions were interpolated onto a 2-km grid and re-run for another 10 years. Finally, the solutions were interpolated onto the 1-km grid and re-run for at least 10 years to establish a statistically steady state, as judged from the time series of domain-integrated total kinetic energy. Daily outputs taken from the final 5 model years are analyzed for this work.

### 2.3 Simulated Flows

In Figures 1a and 1b, we show instantaneous snapshots of the sea surface potential temperature (color contours) upon reaching a statistically steady state along with the model bathymetries (gray sheet) in the “Reference” run and in the “4Mt25Yt” run (four sets of canyon/ridge with onshore/offshore excursion of Yt = 25 km are imposed; see Table 2), respectively. In both simulations, mesoscale eddies are visible throughout the model domain. Selected isopycnal surfaces (θ = 5°C) indicated by the blue sheets incrop onto continental slopes and tilt in the opposite as the seafloor in both cases, corresponding to prograde frontal systems.

In Figures 1c and 1d, we present selected contours of time-/zonal-averages of potential temperature ⟨θ⟩ (dashed gray contours) and along-slope velocity ⟨u⟩ (solid gray contours), superimposed on the logarithm of time-/zonal-mean eddy kinetic energy per unit mass (color), defined as

\[
⟨\text{EKE}⟩ = \frac{1}{2} ⟨u'² + v'²⟩.
\]

Here \(⟨•⟩ = \frac{1}{L_x} \int • \, dx\) represents the zonal-mean operator, \(• = \frac{1}{T_{\text{run}}} \int_{t_0}^{t_0+5 \text{ years}} • \, dt\) denotes the time-mean operator, \([u, v]\) are the zonal and meridional velocity components, and the prime \(•' = • - \bar{•}\) stands for the deviation of a quantity from its time-mean. In both simulations, EKE is surface-intensified and peaks over the upper slope \((y \in [150, 200] \, \text{km})\), where the along-slope flow is strongest across the model domain. A second peak of EKE is visible at \(y \simeq 450 \, \text{km}\) in the “4Mt25Yt” run, induced mainly by a few coherent vortices lasting throughout the analysis period (see Movie S1 and Supporting Information document). In addition, the along-slope flow produced by the “4Mt25Yt” simulation (maximum \(⟨u⟩\) ≃ 0.6 m/s) is weaker than that in the “Reference” run (maximum \(⟨u⟩\) > 0.8 m/s). These differences may be reflecting differing dynamics of prograde fronts induced by topographic canyons and ridges, but are nonetheless inconsequential to our results reported later (cf. Section 4.5). We therefore defer the investigation of the eddy dynamics in prograde fronts over corrugated continental slopes to future work.
3 Scalings for the Cross-Slope Eddy Buoyancy Diffusivity

In this section, we first quantify and characterize the cross-slope eddy buoyancy diffusivity in our eddy-resolving “Smooth” simulations. The diagnosed eddy diffusivity is then used for assessing the skill of extant theories of baroclinic eddy fluxes across prograde fronts. Following this assessment, slope-aware scalings for the eddy buoyancy diffusivity are developed.

3.1 Diagnosed Eddy Buoyancy Diffusivity

As in Wang and Stewart (2020), we focus our attention on the depth-averaged but latitudinally-varying eddy buoyancy diffusivity. This option allows for direct comparisons with previous works that typically focused on bulk quantifications (i.e., regional averages) of cross-slope eddy diffusivities (e.g., Spall, 2004; Isachsen, 2011) and adheres to the horizontally-varying but depth-independent forms of parameterization schemes currently implemented in ocean models (e.g., Griffies et al., 2005; Jansen et al., 2019; Kong & Jansen, 2021; Holmes et al., 2022; Mak et al., 2022).

In Figure 2, we plot the depth-averaged cross-slope eddy buoyancy diffusivity as a function of latitude diagnosed from our reference simulation (black dots), calculated as

\[
K_\theta = -\frac{F_{TE}}{\left\langle \int_{-|H|}^{0} \frac{\partial b}{\partial y} \, dz \right\rangle}, \quad (5a)
\]

\[
F_{TE} = \left\langle \int_{-|H|}^{0} \overline{v'b'} \, dz \right\rangle, \quad (5b)
\]

where \( b = -g(\rho - \rho_0)/\rho_0 \) represents the buoyancy with \( \rho_0 \) denoting the Boussinesq reference density. It should be noted that although \( K_\theta \) is commonly interpreted as the eddy buoyancy diffusivity, it is associated with the “skew flux” of buoyancy, as opposed to the diffusive flux (e.g., Gent, 2011; Abernathey et al., 2013; Kong & Jansen, 2021). In Equations (5a) and (5b), the transient eddy buoyancy flux \( F_{TE} \) and the cross-slope buoyancy gradient are integrated separately in the vertical direction to avoid ill-defined \( K_\theta \) if the cross-slope buoyancy gradient approaches zero at certain depths (Jansen et al., 2015). Notice that the rotational component of the horizontal eddy flux vector (J. Marshall & Shutts, 1981) is automatically excluded from \( F_{TE} \), a result that follows from application of the two-dimensional divergence theorem to a zonally re-entrant channel domain (Wang & Stewart, 2020). Approaching the southern boundary (e.g., \( y < 50 \) km) and seaward of \( y = 360 \) km, \( K_\theta \) becomes much less well-defined and frequently changes sign because the isopycnals are approximately flat throughout the water column. Here we neglect these negative depth-averaged diffusivities, most of which are likely caused by the proximity to the sponge layer or solid boundaries (e.g., Wang & Stewart, 2018).

A salient feature of \( K_\theta \) is that its amplitude decreases substantially from the relatively flat-bottomed regions to the continental slope region (i.e., \( y \in [Y_s - W_s, Y_s - W_s] \)). In particular, \( K_\theta \) is of \( \mathcal{O}(10^3 - 10^4 \text{ m}^2/\text{s}) \) shoreward of \( y = 118 \) km and seaward of \( y = 348 \) km, decays almost monotonically to \( 146 \) m$^2$/s and \( 60 \) m$^2$/s at the shelf break (\( y = 150 \) km) and at the junction of slope to open ocean regions (\( y = 250 \) km), respectively, and reaches its minimum of \( 13 \) m$^2$/s near the mid-slope position (\( y = 197 \) km). This spatial distribution of \( K_\theta \) is consistent with previous findings that a sloping bottom suppresses baroclinic eddy fluxes across prograde fronts (Blumssack & Gierasch, 1972; Mechoso, 1980; Spall, 2004; Isachsen, 2011; Pennel et al., 2012; Brink, 2012; Brink & Cherian, 2013; Poulin et al., 2014; Brink, 2016; Hetland, 2017; Cimoli et al., 2017; Trodahl & Isachsen, 2018; Ghaffari et al., 2018; S.-N. Chen et al., 2020).

It is also of interest to examine the spatial variations of the bulk slope parameter,
\[ \delta = -\left\langle \frac{\partial H}{\partial y} \right\rangle \left( \int_{-|H|}^{0} N^2 \, dz \right) \left/ \left( \int_{-|H|}^{0} \frac{\partial H}{\partial y} \, dz \right) \right\rangle, \tag{6} \]

and the bulk slope Burger number,

\[ S = \left\langle \left| \frac{\partial H}{\partial y} \right| \frac{1}{|H|} \int_{-|H|}^{0} \frac{N}{f_0} \, dz \right\rangle, \tag{7} \]

across our model domain, because both parameters have been used to constrain baroclinic eddy growths and fluxes across prograde fronts (see discussions in Section 1; Pennel et al., 2012; Brink, 2012; Stewart & Thompson, 2013; Brink & Cherian, 2013; Poulin et al., 2014; Brink, 2016; Hetland, 2017; Cimoli et al., 2017; Ghaffari et al., 2018). In Figure 2b, we present the additive inverse of the slope parameter (i.e., \(-\delta\)) and the slope Burger number as functions of latitude in our reference simulation.

The magnitude of \(-\delta\) reaches a well-defined local maximum of 6.5 near the mid-slope position \((y = 219)\) km, and decays monotonically toward the local minima of 1.9 near the shelf break \((y = 157)\) km and of 2.5 in the open ocean \((y = 333)\) km. Throughout this latitudinal range, no direct correlation is found between the eddy diffusivity and the slope parameter, which contrasts with the case of retrograde fronts (see Figure 3a of Wang & Stewart, 2020). However, other dimensional quantities, such as the thermal wind velocity and the slope width, may be combined with analytical functions of \(\delta\) to quantify the baroclinic eddy fluxes (e.g., Spall, 2004). Across the relatively flat-bottomed regions \((y < 70)\) km or \(y > 350)\) km), the isopycnals become approximately flat, reaching a depth-averaged slope of \(\sim O(10^{-5})\) (i.e., approximately three orders of magnitude smaller than the topographic slope), leading to a much less well-defined slope parameter.

Figure 2b shows that \(S\) in the reference run is almost latitudinally symmetric around the mid-slope position, reaching its maximum of approximately 0.8 at \(y = 192\) km and decays monotonically toward the continental shelf and open ocean regions. This spatial symmetry is largely governed by that of the topographic steepness \(\frac{\partial H}{\partial y}\) following (7). “Smooth” simulations with varied configuration parameters from the reference setup (Table 2) yield a range of \(S\) whose maxima reside between 0.5 and 1.5 over the continental slope (not shown). Comparison between Figures 2a and 2b reveals that the eddy buoyancy diffusivity is increasingly suppressed as the amplitude of \(S\) grows larger, consistent with previous studies of prograde shelf fronts (Brink, 2012; Brink & Cherian, 2013; Brink, 2016).

In following subsections, we test the skill of extant approaches to constraining the cross-slope eddy buoyancy diffusivity against our simulations. These approaches generally postulate a scaling relation,

\[ K_\theta \sim F \cdot K_0, \tag{8} \]

where \(F\) denotes an analytical function of \(\delta\) (e.g., Spall, 2004; Stewart & Thompson, 2013) or \(S\) (e.g., Brink, 2012; Brink & Cherian, 2013; Brink, 2016) to account for the topographic influence on baroclinic eddy fluxes, and \(K_0\) denotes a reference diffusivity that is pertinent to open-ocean regions and thus typically independent of the bottom slope. We summarize these scalings in Table 3.

To avoid any influence of lateral boundaries on our model diagnostics (Figure 2), we further confine our analyses to the latitudinal range of \(y \in [50, 350]\) km and neglect negative diffusivities over the continental shelf and in the far open-ocean region (i.e., \(y \geq 350\) km). Such a delineation of analysis regions yields 2629 (1591) latitudinal bands in the “Smooth” (“Corrugated”) simulations at which the scaling-diagnosis comparison for the eddy buoyancy diffusivity can be established.
3.2 Skill of a $\delta$-Dependent Scaling

A representative scaling for $K_\theta$ that depends explicitly on the slope parameter follows from Spall (2004), who parameterized the cross-slope eddy buoyancy fluxes in simulations of convective marginal seas as

$$\mathbf{\nu J} \sim e^{-2|\delta|} \cdot U_{tw} \cdot \Delta \mathbf{b},$$  

(9)

where $U_{tw} = \left< \int_0^H |U| \cdot \frac{\partial \mathbf{b}}{\partial y} \, dz \right>$ stands for the thermal wind velocity of the prograde slope front and $\Delta \mathbf{b}$ represents the mean horizontal buoyancy contrast across the front. The exponential decay function of $|\delta|$ in (9) was constructed to fit the analytical solution of eddy buoyancy fluxes produced by the modified linear Eady (1949) model for the case of $\delta < 0$ (Blumsack & Gierasch, 1972). In the simulations of Spall (2004), the range of $|\delta|$ was confined to within $O(1)$ ($\delta$ was specifically taken to be -0.8, -0.4, or -0.2), which was argued to be typical of the boundary currents in the Nordic Seas (Isachsen, 2011).

Dividing the mean cross-slope buoyancy gradient $\Delta \mathbf{b}/L_f$ on both sides of (9) yields a scaling for the eddy buoyancy diffusivity,

$$K_{S04} = \mathcal{F}_{S04}(\delta) \cdot U_{tw} \cdot L_f,$$  

(10a)

$$\mathcal{F}_{S04}(\delta) = \Gamma_{S04} \cdot e^{-2|\delta|},$$  

(10b)

where $L_f$ denotes the cross-slope frontal width and $\Gamma_{S04}$ represents a non-dimensional coefficient, estimated to be 0.025 by Spall (2004) following Spall and Chapman (1998) and Visbeck et al. (1996).

Specification of the length scale $L_f$ is required to test the utility of $K_{S04}$ against our eddy-resolving simulations. Visbeck et al. (1997) loosely defined $L_f$ as “the width of a baroclinic zone” (see also Green, 1970), across which the local Eady (1949) growth rate (calculated without accounting for a sloping bottom) exceeds 10% of the maximum growth rate across the model domain. Alternatively, $L_f$ can be defined as the slope width, i.e., $L_f \approx 2W_s$ (e.g., Spall, 2004; Stewart & Thompson, 2016). Diagnostics from the reference model run reveal that the latitudinal ranges selected via these two criteria almost overlap one another (not shown), occupying the entire continental slope region. However, even if these variants of $L_f$ in (10) apply to the bulk of eddy fluxes across the steep slope, they are much less relevant for the eddy diffusivity across the shelf or open-ocean regions, which fall outside of the diagnosed baroclinic zones (Visbeck et al., 1997) in our simulations.

We therefore resort to a length scale that can be well-defined for each horizontal location (or latitude) of our model domain. Because the scaling of Spall (2004) was constructed based on the linear theory of baroclinic instability, in which the first Rossby deformation radius is treated as the natural eddy length scale, we define $L_f \equiv L_{d1}$ in (10) following Stone (1972) and plot $K_\theta$ against $K_{S04}$ across our analysis regions of all “Smooth” simulations in Figure 3a. Diagnostics drawn from distinct regions with different degrees of topographic steepness are visualized via the colorbar, which indicates the magnitude of the slope Burger number (Figure 2b). Because the values of $K_\theta$ in our analysis regions span approximately three orders of magnitude (see also Figure 2a), we quantify the predictive skill of each scaling by calculating the Nash and Sutcliffe (1970) coefficient of efficiency, denoted by $R^2$, using the logarithms of diagnosed and theoretically estimated eddy diffusivities (see Appendix A). One should note that $R^2$ thus defined is not equivalent to the squared Pearson correlation coefficient (i.e., the coefficient of determination for linear least-squares regressions), and ranges from $-\infty$ to unity.

No agreement between $K_\theta$ and $K_{S04}$ is found throughout our analysis regions (Figure 3a). Additional inspection of the parameter dependency of $K_{S04}$ reveals the exponential
decay function $F_{S04}$ to be primarily liable for the poor predictive skill of $K_{S04}$. For instance, the diagnosed $K_\theta$ in the reference simulation decreases from 146 m$^2$/s at $y = 150$ km ($|\delta| = 2.1$) to 28 m$^2$/s at $y = 220$ km ($|\delta| = 6.5$), accompanied by a three-order-of-magnitude drop in the value of $F_{S04}$ (Figure 2). Given that the product of the thermal wind velocity and deformation radius varies only by approximately a factor of four between these two locations, the function $F_{S04}$ leads to an over-adjustment to the predicted cross-slope eddy diffusivity. Such an over-adjustment is more conspicuous in Figure 3a, which shows the predicted diffusivity to vary by more than fifteen orders of magnitude. In this case, it is impossible to remedy the scaling (10) by simply re-selecting the eddy velocity or length scales (cf. Section 4.1).

### 3.3 Skill of Extant $S$-Dependent Scalings

A few empirical scalings for $K_\theta$ have also been developed for prograde fronts by previous works (Brink, 2012; Brink & Cherian, 2013; Brink, 2016). These scalings incorporate the slope Burger number $S$ (as opposed to $\delta$) to account for the topographic suppression on the cross-slope eddy buoyancy diffusivity. In this subsection, we assess the skill of these scalings in quantifying the eddy buoyancy diffusivity in our eddy-resolving simulations.

By investigating the eddy dynamics in simulations of inviscid tidal mixing fronts, Brink (2012) formulated a scaling of the eddy buoyancy diffusivity as

$$K_{B12} = F_{B12}(S) \cdot u_e L_d,$$

$$F_{B12}(S) = \frac{\Gamma_{B12}}{C_{B12} \cdot S + 1},$$

where $\Gamma_{B12}$ and $C_{B12}$ are respectively the non-dimensional coefficients measuring the mixing efficiency and the strength of topographic suppression, and $u_e$ stands for the eddy velocity scale. For depth-averaged calculations, the eddy velocity scale can be derived from the depth-averaged EKE, i.e.,

$$u_e = \sqrt{\frac{2}{|H|} \int_{-|H|}^{0} \text{EKE} \, dz}.$$

In the limit of a flat-bottomed ocean, the scaling (11) reduces to the product between the eddy velocity scale and the first Rossby deformation radius, i.e., $K_{B12}(S \rightarrow 0) \sim u_e \cdot L_d$, which resembles the scaling proposed by Cessi (2008) and agrees with the scheme of Eden and Greatbatch (2008) for applications to mid- to high-latitude open-ocean regions, where mesoscale eddies are highly isotropic in the horizontal direction and characterized by a length scale proportional to the deformation radius (Eden, 2007). A key difference between (10) and (11) in the context of a flat-bottomed ocean then arises from the selection of the velocity scale (i.e., $U_{tw}$ versus $u_e$); the dependence on $u_e$ makes $K_{B12}$ constrained by the nonlinear eddy energy budget (Eden, 2007; Cessi, 2008).

As the ocean bed becomes tilted, the eddy diffusivity is predicted to be increasingly suppressed following (11), dictated by the decay function of the slope Burger number $F_{B12}$. A theoretical basis for interpreting the variations of $K_\theta$ with the slope Burger number remains elusive in the literature, but the simulations of Hetland (2017) show that $S$ can be used to remedy the inaccuracy of the baroclinic eddy growth rate predicted by linear theories (Blumsack & Gierasch, 1972) in prograde fronts.

The skill of (11) in quantifying $K_\theta$ across our analysis regions is illustrated in Figure 3b. Here we fix the amplification factor of $S$ in (11), $C_{B12} = 15$, exactly following Brink (2012) (re-tuning this parameter does not qualitatively alter the results below). Nevertheless, we
select the prefactor of $\Gamma_{B12} = 0.50$ to optimize the logarithmic scaling-diagnosis fit (i.e., $R^2$) based on the model diagnostics across our analysis regions. Here and in subsequent analyses, a constant scaling prefactor often requires some re-adjustment in its specific value from that reported in previous studies, though its order of magnitude remains largely unchanged. The readjustment of $\Gamma_{B12}$, for example, may have reflected the difference of the calculated eddy diffusivity in the initial value problems of Brink (2012), which was spatially- and temporally-confined, from that in equilibrated flows. The scaling-diagnosis fit is slightly improved (Figure 3b, $R^2_{\text{total}} = 0.23$) in comparison to that based on the linear Eady (1949) solutions (Figure 3a, $R^2_{\text{total}} < 0$), but $K_\theta$ is underestimated (overestimated) by $K_{B12}$ across relatively flat-bottomed (sloping-bottomed) regions. In Figure 2a, we show the latitudinal variation of $K_{B12}$ in our reference model domain (solid orange curve), Consistent with the scatter plot shown in Fig. 3b, the amplification of $K_{B12}$ from the continental slope toward the open ocean is far more moderate than that of $K_\theta$ (black dots), and $K_{B12}$ even predicts a suppressed eddy diffusivity over the continental shelf. Therefore, $K_{B12}$ fails to capture the transition of $K_\theta$ from the steep slope toward relatively flat-bottomed regions. In a follow-up study, Brink and Cherian (2013) considered the effects of bottom frictional drag and submesoscale symmetric instabilities on prograde frontal eddies, and updated the scaling for $K_\theta$ of Brink (2012) to

$$K_{BC13} = F_{BC13}(S) \cdot u_e L_d,$$

(13a)

$$F_{BC13}(S) = \frac{\Gamma_{BC13}}{C_{BC13} \cdot S^2 + 1}.$$

(13b)

This scaling differs from (11) only in the power of $S$ and the specific values of two constant factors (i.e., $\Gamma_{BC13}$ and $C_{BC13}$). In Figure 3c, we plot $K_{BC13}$ against $K_\theta$ across our analysis regions, with $C_{BC13} = 30$ exactly following Brink and Cherian (2013), and $\Gamma_{BC13} = 0.40$ chosen to optimize the logarithmic scaling-diagnosis fit. The scaling (13) performs slightly better than (11) in capturing the cross-slope variations of $K_\theta$ in our simulations, elevating the overall scaling-diagnosis fit from $R^2_{\text{total}} = 0.23$ to $R^2_{\text{total}} = 0.32$. However, the deviation of the diagnosed diffusivity from the predicted diffusivity remains substantial. Both $K_{B12}$ and $K_{BC13}$ were formulated on the basis of mixing length theory (Prandtl, 1925) with the bulk EKE providing an eddy velocity scale and the Rossby deformation radius serving as the typical mixing length. Using a straightforward dimensional scaling, the eddy mixing length can be further decomposed into an eddy timescale and the eddy velocity scale. This approach was taken by Brink (2016) to construct the scaling for $K_\theta$ across prograde shelf fronts energized by episodic winds:

$$K_{B16} = F_{B16}(S) \cdot \frac{u^2_e}{f_0},$$

(14a)

$$F_{B16}(S) = \frac{\Gamma_{B16}}{C_{B16} \cdot S^2 + 1}.$$

(14b)

Here $\Gamma_{B16}$ and $C_{B16}$ are constant factors, defined below. In (14), the inverse Coriolis frequency is selected as the eddy timescale, and the functional form of the prefactor $F_{B16}$ is identical to $F_{BC13}$ defined in (13). The skill of $K_{B16}$ in quantifying the cross-slope eddy buoyancy diffusivity is shown in Figure 3d, with $C_{B16} = 2.0$ as in Brink (2016) and $\Gamma_{B16} = 2.0$ (taken as 0.30 by Brink, 2016) to optimize the logarithmic scaling-diagnosis fit in our simulations. However, no overall agreement is found between $K_{B16}$ and $K_\theta$ ($R^2_{\text{total}} < 0$), which indicates that the
inverse Coriolis frequency fails to characterize the eddy timescale in equilibrated baroclinic flows.

Our analyses so far reveal the inability of extant scalings to quantify the eddy buoyancy diffusivities across our shelf-to-open-ocean analysis regions. However, the slope Burger number $S$ exhibits a potential to adapt scalings of $K_\theta$ developed for the open ocean to continental shelves and slopes occupied by prograde fronts (Brink, 2012; Brink & Cherian, 2013; Brink, 2016; Hetland, 2017). In what follows, we evaluate scalings that have been broadly validated in open ocean environments (D. P. Marshall et al., 2012; Mak et al., 2017, 2018, 2022; Jansen et al., 2015, 2019; Kong & Jansen, 2021) and then combine these scalings with analytical functions of $S$ following (8) to construct new slope-aware scalings for eddy buoyancy diffusivities across prograde fronts.

### 3.4 The Slope-Aware GEOMETRIC Scaling

A parameterization for $K_\theta$ that has been recently implemented in ocean general circulation models and extensively tested for open ocean environments (Mak et al., 2018, 2022) builds upon the GEOMETRIC framework proposed by D. P. Marshall et al. (2012), who showed that the norm of the Eliassen-Palm eddy stress tensor is strictly bounded by the total eddy energy in QG turbulence, and derived a scaling relation using a down-gradient closure of the eddy buoyancy fluxes that contribute to the eddy stress (see also Maddison & Marshall, 2013; Bachman et al., 2017; Poulsen et al., 2019).

The GEOMETRIC framework specifically formulates the theoretical eddy buoyancy diffusivity as

\[
K_{\text{GEOM}} = \alpha \frac{\sqrt{Ri}}{f_0} E, \tag{15a}
\]

\[
E = \left( \frac{1}{|H|} \int_{-|H|}^{0} \text{EKE} \, dz + \frac{1}{|H|} \sum_{i=1}^{N_\text{lay}-1} \text{EPE}_i \right), \tag{15b}
\]

\[
\text{EPE}_i = \frac{1}{2} g_i \eta_i^{1/2}, \tag{15c}
\]

where $\alpha$ represents a non-dimensional prefactor coefficient with its magnitude bounded by unity, $E$ and EPE are respectively the depth-averaged total eddy energy and the local eddy potential energy per unit mass, and

\[
Ri = \left( \frac{1}{|H|} \right) f_0^2 \left( \int_{-|H|}^{0} N^2 \, dz \right) \left( \int_{-|H|}^{0} \frac{\partial \eta}{\partial y} \, dz \right)^2 \tag{16}
\]

is the bulk Richardson number. In (15b)–(15c), the subscript $i$ denotes the counting of isopycnal layers from surface to bottom with its maximum denoted by $N_\text{lay}$, $\eta_{i+1/2}$ is the height of the isopycnal interface between layer $i$ and $i + 1$, and $g_i = g(\rho_{i+1} - \rho_i)/\rho_0$ stands for the associated reduced gravity. To accurately calculate the EPE, we have defined a sum of $N_\text{lay} = 121$ isopycnal layers and converted the model diagnostics from geopotential coordinates onto isopycnal coordinates following Young (2012). Among the selected isopycnal layers, 71 reside in the buoyancy (i.e., potential temperature) range of $[0, 10] ^\circ C$, with their buoyancy intervals selected based on the prescribed vertical discretization of the buoyancy field in the northern sponge layer, and the other 50 fall into the buoyancy range of $(10, 15] ^\circ C$, with their buoyancy interval fixed as $0.1 ^\circ C$ based on the prescribed linear surface temperature profile.

In Figure 4a, we present the scatter plot of the GEOMETRIC scaling (15) against $K_\theta$ across our analysis regions for all “Smooth” simulations. We select a constant GEOMETRIC prefactor $\alpha = \alpha_0 = 0.07$ to optimize the logarithmic scaling-diagnosis fit for diagnostics...
drawn from shelf and open ocean regions. This value of $\alpha_0$ is close to the value of $\alpha = 0.043$
estimated from an eddy-resolving model of the Antarctic Circumpolar Current (Poulson
et al., 2019). It also resembles the value of $\alpha \approx 0.08$ found in the open ocean regions of
the channel simulations configured with retrograde winds by Wang and Stewart (2020).
Consequently, the GEOMETRIC scaling quantitatively reproduces the eddy buoyancy dif-
fusivities across the relatively flat-bottomed regions in our simulations ($R_{\text{shelf}}^2 = 0.94$ and
$R_{\text{ocean}}^2 = 0.91$). Over continental slopes, however, $K_{\text{GEOM}}$ significantly overestimates the
eddy buoyancy diffusivity ($R_{\text{slope}}^2 < 0$), indicating the necessity of further modifications for
$K_{\text{GEOM}}$ to account for the topographic effects on eddy buoyancy fluxes.

Insights can be gained by quantifying the shelf-to-open-ocean transition of the GEO-
METRIC prefactor. In Figure 4b, we plot the diagnosed GEOMETRIC prefactor,

\begin{equation}
\alpha = K_\theta \left( \frac{\sqrt{R_i}}{f_0} \right), \tag{17}
\end{equation}

normalized by $\alpha_0$ as a function of the slope Burger number $S$, with diagnostics drawn from
the shelf and open ocean regions (continental slope region) marked in blue (orange). The
data converge onto a decay function-like relationship, with the value of $\alpha$ ranging from 0.04
to 0.16 (corresponding to $\alpha/\alpha_0$ ranging from 0.6 to 2.3) for $S \approx 0$ and dropping below 0.007
(corresponding to $\alpha/\alpha_0 < 0.1$) for $S > 0.5$. This quantitative transition of $\alpha$ between the
relatively flat- and sloping-bottomed regions can be captured by many empirical functions
of $S$, among which we opt for

\begin{equation}
\alpha = \alpha_0 \cdot F_{\text{GEOM}}(S), \tag{18a}
\end{equation}

\begin{equation}
F_{\text{GEOM}}(S) = \frac{1}{\mu_1 \cdot S^{\mu_2} + 1}, \tag{18b}
\end{equation}

as a trade-off between simplicity and accuracy. Here $\mu_1 = 7.8$ and $\mu_2 = 1.5$ are two non-
dimensional constants chosen to optimize the functional dependence of $\alpha$ on $S$ across the
continental slope (optimizing this function across the entire analysis region leads to nearly
identical results; not shown). The reformulated GEOMETRIC prefactor (18) (normalized
by $\alpha_0$) is plotted in Figure 4b using a black curve. Following (18), $\alpha$ approaches $\alpha_0 = 0.07$
for a flat-bottomed region (i.e., $S \approx 0$; see Figure 4a), and monotonically decays toward
zero over increasingly steep slopes.

As shown in Figure 4c, adoption of the $S$-dependent prefactor (18) in the GEOMETRIC
scaling for the eddy diffusivity (15) substantially improves the scaling-diagnosis fit across
the continental slope ($R_{\text{slope}}^2 = 0.73$; note the largely unaltered scaling-diagnosis fits for shelf
and open ocean regions), and the scattered diagnostics now collapse onto a straight line on
the logarithm-scale axes ($R_{\text{total}}^2 = 0.94$).

Following previous authors who focused on parameterizing the regionally-averaged eddy
buoyancy diffusivity across a large-scale baroclinic zone (e.g., Visbeck et al., 1997; Spall,
2004; Jansen et al., 2015; Stewart & Thompson, 2016; Kong & Jansen, 2021), we further
show in Figure 4d the relationship between the regionally-averaged “slope-aware” form of
$K_{\text{GEOM}}$ and $K_\theta$ across the continental shelf (circles), the continental slope (dots), and the
open ocean (triangles) in our analysis regions. As expected, $K_{\text{GEOM}}$ constructed with the
$S$-dependent prefactor $\alpha = \alpha_0 \cdot F_{\text{GEOM}}$ quantitatively reproduces the bulk eddy buoyancy
diffusivities in all three regions ($R^2 = 0.97$).

The spatial variations of these GEOMETRIC-parameterized eddy diffusivities in our
reference simulation are highlighted in Figure 2a. Treating the GEOMETRIC prefactor as
a constant ($\alpha = \alpha_0 = 0.07$), the GEOMETRIC scaling (blue curve) quantitatively captures
the latitudinal structure of $K_\theta$ (black dots) over the shelf and in the open ocean, but
overestimates $K_\phi$ by up to one order of magnitude over the slope (e.g., $K_{\text{GEOM}}\big|_{a=\alpha_0} = 141.8 \text{ m}^2/\text{s}$ versus $K_\phi = 13.4 \text{ m}^2/\text{s}$ at $y = 200 \text{ km}$). By contrast, replacing the constant prefactor with that depending on $S$ following (18) results in the GEOMETRIC scaling (green curve) much more closely adhering to the diagnosed eddy diffusivity over the steep slope (e.g., $K_{\text{GEOM}}\big|_{a=\alpha_0, S_{\text{GEOM}}(S)} = 23.8 \text{ m}^2/\text{s}$ versus $K_\phi = 13.4 \text{ m}^2/\text{s}$ at $y = 200 \text{ km}$).

We note that there are sharp spatial variations in the eddy diffusivities predicted by the GEOMETRIC scaling over the continental shelf and in the open ocean (Figure 2a, blue and green curves). These variations are mainly caused by the vanishing horizontal buoyancy gradient (i.e., $\partial_y \bar{b}$ approaching zero) in these regions, which may easily amplify the calculated Richardson number upon which the the GEOMETRIC scaling depends. However, these spikes in $K_{\text{GEOM}}$, which emerge on length scales of ~10 km, do not necessarily imply that spatially-discontinuous diffusivities would result if $K_{\text{GEOM}}$ were implemented prognostically in ocean climate models. Indeed, these models typically adopt a horizontal grid spacing of e.g., 25 km or coarser, and implement certain numerically-motivated constraints (e.g., spatial smoothing, and upper/lower bounds for the predicted diffusivity) to avoid excessively large parameterized eddy fluxes that could lead to numerical instabilities (e.g., Mak et al., 2018). Additional work is needed to investigate the prognostic performance of $K_{\text{GEOM}}$ using coarse-resolution simulations of continental slope flows, but is beyond the scope of this study.

### 3.5 The Slope-Aware Mixing Length Theory-Based Scaling

The GEOMETRIC framework builds an explicit linear scaling relation between the eddy energy and the eddy buoyancy diffusivity, similar to the one defined by Brink (2016) (see Equation (14)). Alternatively, one may formulate the eddy diffusivity as the product of an eddy velocity scale and an eddy length scale following the mixing length theory (Prandtl, 1925), as proposed by Brink (2012) and Brink and Cherian (2013) (see Equations (11)–(13)). While depending explicitly on the eddy velocity scale $u_e$, and thus the nonlinear eddy energy budgets, the scalings of Brink (2012) and Brink and Cherian (2013) incorporate the first Rossby deformation radius as the eddy mixing length. The first Rossby deformation radius directly reflects the length scale of linearly unstable baroclinic modes (Vallis, 2017), which may become less useful for parameterization purposes if baroclinic instabilities lead to mesoscale turbulence that supports an inverse cascade of energy (e.g., Larichev & Held, 1995). In this case, the length scale of the eddy field may become independent of the linear processes, and the eddy fluxes may be limited by different length scales pertinent to the nonlinear eddy field (see discussions below; Thompson & Young, 2007; Eden & Greatbatch, 2008; Jansen et al., 2015).

Strategies for inferring the length scales that limit nonlinear eddy fluxes were discussed by Jansen et al. (2015, 2019). Specifically, if one considers the length scale at which the inverse barotropic energy cascade is arrested, the topographic Rhines scale,

$$L_{Rh} = \sqrt{u_e/\beta_t},$$

arises as a candidate for the mixing length over sloping ocean beds (Vallis & Maltrud, 1993; Pringle, 2001; Thompson, 2010; Stewart & Thompson, 2016; Brink, 2017; Kong & Jansen, 2021), where $\beta_t = \left\langle \left( \frac{\partial H}{\partial y} \right) \right\rangle$ denotes the topographic PV gradient. In the absence of topographic PV arresting effect, one may assume that the production rate of eddy energy via baroclinic instabilities balances the inverse cascade rate of eddy energy and that the eddy mixing length matches the length scale at which the barotropic EKE spectrum peaks. These ideas motivate Larichev and Held (1995) and Jansen et al. (2015) to derive the Eady scale (this nomenclature is borrowed from Kong & Jansen, 2021),

$$L_E = u_e \sigma_e^{-1} = u_e \sqrt{\frac{Ri}{f_0}},$$

(20)
where $\sigma_E = f_0/\sqrt{R_i}$ denotes the Eady (1949) growth rate. Notice that both $L_{Rh}$ and $L_E$ have been defined using the depth-averaged EKE following Kong and Jansen (2021), as opposed to using the barotropic EKE (i.e., the EKE calculated from instantaneous depth-averaged velocities) for a multiple-layer or continuously stratified ocean (e.g., Cessi, 2008; Jansen et al., 2015). The ratio of the barotropic EKE to the depth-averaged EKE in our “Smooth” simulations varies from a maximum of $\sim 0.9$ over the shelf to a minimum of $-0.4$ over the steep slope (not shown). Therefore, adopting the barotropic EKE for parameterizing $K_\theta$, which varies across four orders of magnitude in our analysis regions, only leads to slight quantitative changes in the results reported below.

Figures 5a and 5b show the scatter plots of the mixing length theory-based scaling,

$$K_{MLT} = \Gamma \cdot u_e L_{mix},$$

against $K_\theta$ diagnosed across the analysis regions of all “Smooth” simulations, in which the eddy mixing length $L_{mix}$ is cast as $L_{Rh}$ and $L_E$, respectively. In both cases, constant prefactor coefficients are selected ($\Gamma = 0.17$ for $L_{mix} = L_{Rh}$ and $\Gamma = 0.08$ for $L_{mix} = L_E$) such that the logarithm-based scaling-diagnosis misfits are minimized across continental shelf and open ocean regions.

Figure 5a reveals that the Rhines scale-based scaling

$$K^{Rh}_{MLT} = \Gamma \cdot u_e L_{Rh}$$

has limited skill in quantifying the eddy buoyancy diffusivity. Specifically, $K^{Rh}_{MLT}$ overestimates (underestimates) $K_\theta$ over continental slopes (in continental shelf and open ocean regions), as highlighted by the red (blue) dots primarily residing below (above) the black line in Figure 5a. Quantitatively consistent results are shown by Figure 2a, in which the spatial variation of $K^{Rh}_{MLT}$ in our reference simulation (dashed orange curve) is much less sensitive to the presence of sloping topography than that of the diagnosed eddy diffusivity (black dots). These analyses indicate that the bathymetric suppression of cross-slope eddy buoyancy fluxes should be measured by quantities other than the topographic Rhines scale.

Comparison between Figures 5a and 5b reveals that the Eady scale-based scaling,

$$K^{E}_{MLT} = \Gamma \cdot u_e L_E = \Gamma \frac{\sqrt{R_i}}{f_0} \left\langle \int_0^H \frac{2}{|H|} \left| EKE \right| d_\zeta \right\rangle,$$

provides a quantitatively better prediction of $K_\theta$ across continental shelf ($R^2_{shelf} = 0.94$) and open ocean ($R^2_{ocean} = 0.92$) regions than the Rhines scale-based scaling ($R^2_{shelf} = 0.28$ and $R^2_{ocean} = 0.57$), consistent with the finding of Jansen et al. (2015) that the Eady scale serves as the optimal eddy mixing length for an $f$-plane, flat-bottomed ocean. However, as with the Rhines scale-based scaling, $K^{E}_{MLT}$ significantly overestimates the eddy buoyancy diffusivity across continental slopes ($R^2_{slope} < 0$).

To adapt $K^{E}_{MLT}$ to continental slopes, we note that the Eady scale-based scaling (23), though building upon different theoretical arguments (Larichev & Held, 1995), is parametrically similar to the GEOMETRIC scaling (15). Indeed, these two scalings differ only in the eddy energy type adopted ($K^{E}_{MLT}$ adopts the EKE whereas $K^{GEOM}$ adopts the total eddy energy). In our simulations, the depth-averaged EPE is almost linearly proportional to the depth-averaged EKE at each latitude ($R^2 = 0.91$, not shown), which suggests that the approach to reconstructing the GEOMETRIC scaling over sloping topography (e.g., Equations (17)–(18)) may readily apply to the reconstruction of the Eady scale-based scaling.

These considerations motivate us to reformulate the Eady scale-based scaling as
where $\epsilon_1 = 6.7$ and $\epsilon_2 = 1.4$ are two non-dimensional constants chosen to optimize the functional dependence of $K_\theta / (\Gamma u_e L_E)$ on $S$ across continental slopes in our simulations, and $\Gamma = 0.08$ retains its value as in Figure 5b.

In Figures 5c and 5d, we show that the slope-aware Eady scale-based scaling (24) is capable of quantifying the latitudinally-varying ($R_{\text{total}}^2 = 0.94$) as well as regionally-averaged ($R_{\text{total}}^2 = 0.97$) eddy diffusivities in our “Smooth” simulations, with the scaling-diagnosis scatter patterns nearly identical to those shown in Figures 4c and 4d, respectively.

4 Discussion

4.1 Limitations of $\delta$-Dependent Scalings

In Section 3.2, we show that the scaling for $K_\theta$ proposed by Spall (2004), or more specifically, the exponential decay function of $|\delta|$, fails to constrain the diagnosed eddy diffusivity in our suite of simulations. This may not be as surprising as it seems for several reasons:

(i) The scaling (9) was derived directly from the linear solutions of the modified Eady (1949) model (Blumsack & Gierasch, 1972), which is limited in scope due to the neglect of interior PV gradients and its instability to capture nonlinear eddy features resulting from turbulent cascade (Troccoli & Isachsen, 2018).

(ii) The primitive equation simulations of Isachsen (2011) have shown that the normalized eddy buoyancy diffusivity $K_\theta / (U_{\text{tw}} L_d)$ for thermally-driven slope fronts can vary by at least one order of magnitude even with fixed values of $\delta < 0$, which indicates that $\delta$ may not be the only topographic parameter that governs the quantitative variations of $K_\theta$, particularly when the ocean depth variation is comparable to the mean depth itself (Hetland, 2017). Indeed, the range of $\delta$ chosen by Spall (2004) might be too confined to verify the efficacy of (9) across a broader range of parameter space (Figure 2b).

(iii) In parallel, S.-N. Chen et al. (2020) showed that the prediction of Blumsack and Gierasch (1972) deteriorates as $|\delta|$ increases in simulations of prograde shelf fronts, suggestive of missing dynamics (e.g., lateral shear) from the modified Eady (1949) model.

(iv) In a follow-up study, Spall (2010b) suggested that a thermally-driven boundary current tends to conserve its baroclinic transport such that the value of $\delta$ is maintained as the bottom slope varies, which precludes $\delta$ as a control parameter for baroclinic instabilities of prograde currents. However, it remains difficult to reconcile the reported efficacy of (9) by Spall (2004) and the findings in Spall (2010b) (see also the discussions by Isachsen, 2011).

Here we reiterate that re-selecting the eddy velocity scale (e.g., by replacing $U_{\text{tw}}$ with $u_e$) or length scale (e.g., by defining $L_f$ as $L_{Ri}$ or $L_E$) in (10) does not compensate for the over-adjustment to the theoretical eddy diffusivity due to the exponential decay of $|\delta|$.

We also considered whether the $\delta$-dependent scalings developed for retrograde slope fronts (i.e., $\delta > 0$) by Wang and Stewart (2020) would quantify the cross-slope diffusivities in prograde fronts, provided that the positive definiteness of $\delta$ in these scalings is retained. However, additional calculations (not shown) revealed little utility of these scalings for constraining the eddy buoyancy fluxes across prograde slope fronts in our simulations. In
fact, no satisfactory functional fit of the diagnosed prefactors of scalings examined in this study to the bulk slope parameter was identified.

4.2 Comparisons between \( S \)-Dependent Scalings

Comparison between the \( S \)-dependent scalings examined in Section 3.3 and those proposed in Sections 3.4–3.5 reveals analogies in mathematical formulations. For instance, the scalings proposed by Brink (2012), Brink and Cherian (2013), Brink (2016), and the \( S \)-dependent GEOMETRIC scaling have incorporated comparable decay functions of \( S \), with the main difference residing in the power of \( S \) (see Equations (11b), (13b), (14b) and (18b)).

One may then wonder whether these functions can be swapped without altering the predictive skill of the GEOMETRIC scaling. To address this question, we attempted to recast the GEOMETRIC prefactor as \( \alpha = \alpha_0^c \beta_{B12} \cdot \gamma_{B12}^c, \alpha_0^c / \beta_{BC13} \cdot \gamma_{BC13}^c, \) or \( \alpha_0^c / \beta_{BC16} \cdot \gamma_{BC16}^c \). The reconstructed GEOMETRIC scaling fails to capture the spatial variations of \( K_\theta \) over continental slopes in our analysis regions (\( R_{\text{slope}}^2 < 0.15 \), not shown).

Re-tuning the constant factors other than the power of \( S \) in \( F_{B12}(S), F_{BC13}(S), \) or \( F_{B16}(S) \) (or equivalently, setting \( \mu_2 = 1 \) or 2 but re-tuning the magnitude of \( \mu_1 \) in \( F_{\text{GEOM}}(S) \)) would yield an empirical function whose profile resembles that of \( \alpha_0^c \cdot F_{\text{GEOM}}(S) \) (not shown). The thus reconstructed GEOMETRIC scalings and their skill in quantifying the cross-slope eddy diffusivities produced by the “Smooth” simulations are documented in Appendix B. These scalings do not offer a quantitatively better prediction for \( K_\theta \) than the one proposed in Section 3.4, particularly over continental slopes (e.g., \( R_{\text{slope}}^2 \leq 0.70 \)). However, the analogous scaling-diagnosis fits (Figure B1, Appendix B) to that shown in Figure 4c suggest that the efficacy of \( F_{\text{GEOM}} \) defined by (18) in quantifying the topographic suppression on baroclinic eddy fluxes can be maintained if the power of \( S \) is perturbed (e.g., by varying the magnitude of \( \mu_2 \) across the range of \([1, 2]\)) and the other constant factor \( \mu_1 \) is re-adjusted accordingly.

A related question arises from the comparison between the performances of \( R_{\text{MLT}} \) (Figure 5b) and \( K_{B16} \) (Figure 3d) in quantifying \( K_\theta \) across relatively flat-bottomed regions (blue dots). In this case (\( S \approx 0 \)), the two scalings differ only by a factor of \( \sqrt{R_{\text{M}}} \), and one may then ask: why should a scaling for \( K_\theta \) include the square root of the Richardson number? In Section 4.4, we show that the Eady time scale \( \sqrt{E} \) (see Equation (20)), as opposed to the inertial time scale \( f_0^{-1} \) (see Equation (14)), is more relevant to the equilibrated mesoscale turbulence across our simulations. In our analysis regions with \( S < 0.1 \), the diagnosed \( \sqrt{R_{\text{M}}} \) varies across three orders of magnitude and can thus qualitatively modulate the sensitivity of a scaling to the simulated mean flow stratification.

4.3 Limitations of the Topographic Rhines Scale

In Section 3.5, we show that the topographic Rhines scale is an inappropriate length scale for constraining eddy buoyancy fluxes across prograde shelf and slope fronts. This seems to be at odds with the findings of Stewart and Thompson (2016) that the scaling (22) quantitatively reproduces the bulk eddy thickness diffusivity across the Antarctic Slope Front. One explanation for this discrepancy is that the mesoscale turbulence in the simulations of Stewart and Thompson (2016) possesses a substantial barotropic component, energized by instabilities of an abyssal overflow (see their Figure 8 and related discussions), but is primarily baroclinic over continental slopes in our simulations (as judged from the barotropic to depth-averaged EKE ratio; cf. Section 3.5). The topographic Rhines scale as defined in (19), which takes a barotropic form, may therefore be more relevant to eddy fluxes across the Antarctic Slope Front.

A possible remedy for the Rhines scale-based scaling to account for the baroclinicity of mesoscale turbulence is to use the baroclinic Rhines scale (Williams & Kelsall, 2015; Klocker et al., 2016), \( L_{\text{Rh,.bc}} = \sqrt{\frac{u_{\text{bc}}}{\mu_2 f_0^{-1} \beta}} \), as the mixing length in (21). This length scale is derived
by equating the baroclinic, as opposed to the barotropic, Rossby wave phase speed and the eddy velocity scale. However, a scaling that incorporates $L_{\text{Rh, bc}}$ still fails to capture the latitudinal variation of $K_\theta$ across the continental slope in our “Smooth” simulations (not shown).

Here we note that both the barotropic and baroclinic forms of the Rhines scale were derived for flows controlled by the planetary PV gradient throughout the water column (Rhines, 1975; Williams & Kelsall, 2015). The extent to which a topographic slope may exert a barotropic effect on baroclinic turbulence remains elusive. Therefore, the topographic Rhines scale defined either in a barotropic or baroclinic form could be less relevant to the mesoscale eddy fluxes in a stratified ocean.

### 4.4 Interpretation of the Dependence of $K_{\text{GEOM}}$ on $S$

In Section 3.4, we have shown that the GEOMETRIC scaling (15) can be adapted to continental slopes occupied by prograde fronts via the incorporation of the slope Burger number, which serves as the key control parameter for quantifying the topographic suppression of eddy buoyancy fluxes. The idea of using the slope Burger number to constrain baroclinic eddy behaviors is common in studies of buoyant shelf flows (e.g., Brink, 2012; Brink & Cherian, 2013; Brink, 2016; Hetland, 2017; S.-N. Chen et al., 2020). As with these studies, our approach to constructing the “slope-aware” GEOMETRIC scaling following (18) is highly empirical. In this section, we provide some physical insights into the dependence of $K_{\text{GEOM}}$ on the slope Burger number.

Under the GEOMETRIC framework of eddy parameterizations, the baroclinic eddy energy production rate can be quantified as (see Equation (27) of D. P. Marshall et al. (2012), and Equation (4) of D. P. Marshall et al. (2017)),

$$ \frac{f_0^2}{R_i} K_{\text{GEOM}} = \alpha \cdot \sigma_E \cdot E \equiv 2 \sigma \cdot E, \quad (25) $$

where $\sigma$ represents a growth rate for e.g., the eddy velocity amplitude and is linked to the Eady (1949) growth rate $\sigma_E = \frac{f_0}{\sqrt{R_i}}$. For the linear Eady (1949) model, the most unstable linear wave has a growth rate of $\sigma_{\text{Linear Eady}} = 0.31 \cdot \sigma_E$ (Vallis, 2017), corresponding to $\alpha = 0.62$ in (25). The factor of two in (25) sources from the quadratic form of eddy energy quantities (e.g., Bretherton, 1966).

In the context of a slumping tidal mixing front, Brink (2012) identified the normalized growth rate of baroclinic eddies to be $\sigma / \sigma_E \sim (1 + \mu \cdot S)^{-1}$, where $\mu$ denotes an empirical constant. The scaling of Brink (2012) for eddy growth rate resembles that of Hetland (2017), who found that the linear growth rate derived from the modified Eady (1949) model (e.g., Blumsack & Gierasch, 1972; Spall, 2004; Isachsen, 2011) must be reduced by a factor of $S^{1/2}$ to match the growth rate of baroclinic eddies diagnosed from simulations of shelf buoyant plumes. These empirical findings suggest that the slope Burger number directly controls the baroclinic eddy growth rate in prograde fronts. Invoking Equations (18) and (25), we hypothesize that the eddy growth rate across a prograde slope front scales as

$$ \sigma / [0.31 \cdot \sigma_E] \sim F_{\text{GEOM}} = \frac{1}{\mu_1 \cdot S^{v_2} + 1}. \quad (26) $$

Validation of the scaling relation (26) for equilibrated flows is non-trivial. If one conducts the linear stability analysis of primitive equations for the flow field across the entire model domain (e.g., Lozier & Reed, 2005; Brink, 2006, 2012), one would at best get one solution of the fastest linear growth rate per simulation, which conceals the spatial variation of eddy properties. Another option is to “manually” delineate the flow into several parts in the cross-slope direction and then perform the linear analysis for the bulk of each latitudinal range, but any such delineation for performing the linear analysis is physically unjustified.
Instead, following previous authors focusing on prograde slope fronts (Isachsen, 2011, 2015; Trodahl & Isachsen, 2018), we resort to QG linear stability analysis for each water column across our analysis regions, the procedures of which are detailed in Section 4.4 of Wang and Stewart (2018). Extracting the QG solutions from non-QG flows carries certain caveats, but has previously proven successful in qualitatively understanding the eddy behaviors across steep shelves and slopes (e.g., Poulin et al., 2014; Isachsen, 2015).

In Figure 6a, we plot the fastest growth rate of QG linear mode normalized by the Eady (1949) solution $\sigma/[0.31\sigma_E]$ as a function of the slope Burger number for each water column across our analysis regions, with diagnostics drawn from the slope region (shelf and open ocean regions) marked in orange (blue). Amidst substantial scatter, the data trace out a decay functional shape. Specifically, the scatter in the normalized growth rate becomes centered around unity for continental shelf and open ocean regions where the seafloor is relatively flat (the median of the normalized growth rate is 1.00 for $S < 0.05$). Over continental slopes, the normalized growth rate exhibits a drop toward values below 0.5.

The cross-slope transition of the QG solution is also highlighted in the probability density function diagram of the normalized growth rate in Figure 6b. Strong topographic suppression on the QG wave growth is visible from the distributions of orange bars confined to the left end of the probability density function diagram, yielding a median of 0.23. By contrast, the growth rate exhibits a much broader range of magnitude, with a median of 0.87, once the slope Burger number drops below $\sim 0.4$.

To examine the parameter dependence of the QG linear wave growth rate, we plot the scaling relation (26) in Figure 6a via the black curve, which resembles the functional profile of $f_{\text{GEOMET}}(S)$ shown in Figure 4b. The scaling qualitatively captures the variation of the normalized QG growth rate across our analysis regions, and reduces to the linear Eady (1949) solution in the limit $S \rightarrow 0$.

One may realize that the constant factor of $\alpha_0 = 0.07$, originally defined in (18), has been neglected in (26). This is because linear stability analysis offers no information about the finite amplitude of a fully-developed eddy field, and theoretical estimates may therefore be subject to unknown constants (Isachsen, 2011). Moreover, the QG-based linear wave growth rate often represents an upper bound for the actual eddy growth rate in fully-developed baroclinic turbulence. Indeed, Bachman et al. (2017) identified the GEOMETRIC prefactor as $\alpha = 0.2$ in a set of flat-bottomed, nonlinear Eady-like simulations, indicating an eddy growth rate reduced by a factor of $\sim 3$ compared to the case of the linear Eady (1949) model. Moreover, as discussed in Section 3.4, the eddy growth rate in primitive equation simulations of open ocean flows, as inferred from the diagnosed GEOMETRIC scaling prefactor, is typically one order of magnitude smaller than that in the linear Eady (1949) model (e.g., Poulsen et al., 2019).

Therefore, we argue that the scaling relations (25)–(26) combined with the diagnosed QG linear wave growth rate shown in Figure 6 support our hypothesis. That is, the slope Burger number controls the eddy growth rate across prograde fronts (e.g., Brink, 2012), which further modulates the GEOMETRIC scaling for the cross-slope eddy buoyancy diffusivity.

4.5 Impact of Along-Slope Topographic Variations

As with many previous studies (e.g., Isachsen, 2011; Poulin et al., 2014; Hetland, 2017; Ghaffari et al., 2018; S.-N. Chen et al., 2020), our analyses so far have focused on prograde fronts over shelves and slopes with no bathymetric variation in the along-slope direction. This contrasts with the observation that topographic canyons and ridges are ubiquitous along continental margins (cf. Section 2.2; Harris & Whiteway, 2011). In this subsection, we investigate the extent to which our proposed slope-aware GEOMETRIC and Eady scale-
based scalings for the cross-slope eddy buoyancy diffusivity, summarized in Table 3, apply
to corrugated continental shelves and slopes.

In the presence of zonal topographic corrugations, standing eddies can arise and con-
tribute to the total meridional eddy buoyancy fluxes (Thompson & Naveira Garabato, 2014;
Abernathey & Cessi, 2014; Stewart & Hogg, 2017; Bai et al., 2021; Si et al., 2022), yet
the eddy buoyancy diffusivity $K_\theta$ defined by (5) only accounts for transient eddy fluxes.
Notwithstanding these issues, Si et al. (2022) showed that the vertical eddy fluxes of pro-
grade momentum, determined by the cross-slope buoyancy fluxes (Greatbatch & Lamb,
1990), are not dominated by standing eddies in the prograde Antarctic Slope Front. To
assess this, we calculate the standing eddy fluxes in our “Corrugated” simulations defined
by
$$\left\langle \int_{-|H|}^{0} \tau^i \overline{\theta^i} \, dz \right\rangle,$$
where $\overline{\cdot} = \cdot - \langle \cdot \rangle$ represents the deviation from the zonal-mean (e.g.,
Bischoff & Thompson, 2014). In line with the findings of Si et al. (2022), transient eddy
fluxes are larger than standing eddy fluxes by a factor of at least eight (not shown), which
indicates that all our results supported by “Smooth” simulations should readily apply to
“Corrugated” runs.

Figure 7a presents the scatter plot of $K_\theta$ against the slope-aware form of $K_{GEOM}$ at
each latitude across the analysis regions in all “Corrugated” simulations. The values of
constant factors defined in (18) (i.e., $\alpha_0 = 0.07$, $\mu_1 = 7.8$, and $\mu_2 = 1.5$), which were em-
pirically derived using the “Smooth” simulations, are retained here. In addition, key lateral
vector quantities, including the eddy buoyancy fluxes, the mean buoyancy gradient, and
the bathymetry gradient, exactly follow those defined in Section 3 (i.e., only the merid-
ional components of these vector fields are accounted for). The scaling-diagnosis fits for the
meridional eddy diffusivity remain adequately high (e.g., $R^2_{\text{slope}} = 0.56$ and $R^2_{\text{total}} = 0.94$
for “Corrugated” simulations.

In Figure 7b, we show the scatter plot of the eddy diffusivity against the slope-aware
GEOMETRIC scaling, both of which are averaged through the shelf (circles), slope (dots),
and open ocean (triangles) fractions in our “Corrugated” analysis regions. The overall
scaling-diagnosis fit reaches $R^2 = 0.96$, similar to that shown in Figure 4d for “Smooth”
simulations. These results suggest that our proposed slope-aware GEOMETRIC scaling
is quantitatively robust for constraining eddy buoyancy fluxes across prograde fronts over
complex sloping topography.

As shown in Figures 7c and 7d, the same conclusion can be made regarding the skill
of the slope-aware Eady scale-based scaling in quantifying the eddy buoyancy fluxes across
corrugated continental shelves and slopes. Indeed, the slope-aware Eady scale-based scal-
ing defined in (24) produces nearly identical scatter patterns against the diagnosed eddy
diffusivities to those shown in Figures 7a and 7b.

5 Conclusion and Future Work

In this study, the characteristics and parameter dependencies of the eddy buoyancy
diffusivity across prograde fronts over continental shelves and slopes are quantified using
a suite of eddy-resolving process simulations. Previously proposed scalings for the eddy
buoyancy diffusivity across prograde fronts are tested against our simulations, upon which a
pair of slope-aware scalings capable of reproducing the diagnosed diffusivity are developed.
The main conclusions of this work are as follows:

(i) The eddy buoyancy diffusivity varies by approximately three orders of magnitude
from the continental shelf toward the open ocean regions, and is suppressed where
the bottom slope is steepened (Figure 2), consistent with previous findings of the
suppression effects of the sloping seafloor on baroclinic eddy fluxes (Isachsen, 2011;
Hetland, 2017).
(ii) Scalings based on QG linear theories (e.g., $K_{SO4}$ defined in Equation (10)), which treat the bulk ratio between the topographic and isopycnal slopes as the key control parameter for capturing the topographic suppression on baroclinic eddy fluxes (Blumsack & Gierasch, 1972; Mechosos, 1980; Spall, 2004; Cimoli et al., 2017), are limited in quantifying the cross-slope eddy buoyancy diffusivity across prograde fronts (Figure 3a).

(iii) Scalings depending on the mesoscale eddy energy and the slope Burger number (e.g., $K_{B12}$ in Equation (11), $K_{BC13}$ in Equation (13), and $K_{B16}$ in Equation (14)) (Brink, 2012; Brink & Cherian, 2013; Brink, 2016) can perform quantitatively better than linear theory-based scalings in capturing the eddy buoyancy diffusivity across prograde fronts (Figures 3b and 3d), though significant scaling-diagnosis mismatches remain.

(iv) The GEOMETRIC scheme (15) developed by D. P. Marshall et al. (2012) can be adapted to continental slopes occupied by prograde fronts via the incorporation of the slope Burger number into its scaling prefactor coefficient (Equation (18)). The reformulated, slope-aware GEOMETRIC scaling quantitatively reproduces the diagnosed eddy buoyancy diffusivity from the shelf to the open ocean environments in our simulations (Figures 2a, 4c, 4d, 7a and 7b).

(v) Because of the strong linear correlation between the depth-averaged EKE and EPE (see also Bachman et al., 2017; Wang & Stewart, 2020), our approach to adapting the GEOMETRIC scaling to steep slopes readily applies to the reconstruction of the Eady scale-based scaling (Equation (24)). This means that for an ocean model that maintains either the GEOMETRIC scheme (e.g., MITgcm and NEMO, Mak et al., 2018, 2022) or the mixing length scheme incorporating the Eady scale (e.g., MOM6, Jansen et al., 2019; Kong & Jansen, 2021), an prognostic test of the corresponding slope-aware scaling is feasible.

Since our proposed slope-aware scalings and particularly the relevant tuning parameters entering the empirical $S$-functions are derived from diagnostics of eddy-resolving simulations, prognostic calculations using coarse-resolution runs are essential to examine the robustness of these scalings. For practical implementations of the slope-aware modifications into numerical models, the model should at least resolve the large-scale topography across continental margins such that the key physical parameters in the slope-aware GEOMETRIC or Eady scale-based scalings, including the slope Burger number, can be readily quantified. Further work is required to fully understand the mesoscale eddy energy budgets over continental shelves and slopes, upon which the slope-aware GEOMETRIC and Eady scale-based scalings can be numerically implemented following the approaches of Mak et al. (2018, 2022), Jansen et al. (2019), and Kong and Jansen (2021). Indeed, over continental slopes, EKE can be converted into large-scale potential energy in topographically-rectified flows (Wang & Stewart, 2018), a process yet to be well-constrained for use in existing closures of subgrid-scale EKE. Moreover, the eddy dissipation time scale remains highly uncertain even for the open ocean environment (Mak et al., 2022).

The focus of this work is on prograde fronts, such as those found in the Nordic Seas (Isachsen, 2015) and along the Mid-Atlantic Bight (Lozier & Reed, 2005). This parameter regime complements that studied by Wang and Stewart (2020), who specifically focused on parameterizing eddy buoyancy diffusivities across retrograde fronts. The GEOMETRIC parameterization has also been adapted to retrograde slope fronts by Wang and Stewart (2020) by incorporating the slope parameter (as opposed to the slope Burger number) into the GEOMETRIC prefactor. How such discrepancies in the key control parameters for the eddy buoyancy fluxes across retrograde versus prograde fronts arise remains unclear to the authors, although both linear stability analysis (e.g., Blumsack & Gierasch, 1972) and nonlinear primitive equation simulations (e.g., Isachsen, 2011) have shown that the eddy properties and eddy-induced fluxes across retrograde fronts substantially differ from those across prograde fronts.

As with many previous authors (Visbeck et al., 1997; Jansen et al., 2015, 2019; Mak et al., 2017, 2018), we have focused our attention on the quantification and parameterization of
the bulk (depth-averaged) eddy buoyancy diffusivities. However, the magnitude of the eddy buoyancy diffusivity can vary substantially in the vertical direction across the open ocean (Poulsen et al., 2019; Kenigson et al., 2021) and over the continental slope (Isachsen, 2011; Wang & Stewart, 2020). Given that very few approaches to parameterizing the vertically-varying eddy buoyancy diffusivities are available for use in ocean climate models (e.g., see Section 3 of Kong & Jansen, 2021), an in-depth investigation of the diffusivity vertical structures, particularly over sloping topography, is warranted.

Although our proposed variants of the GEOMETRIC scaling and the Eady scale-based scaling quantitatively reproduce the eddy buoyancy diffusivity across the continental shelf, continental slope, and open ocean environments, the idealized configuration of our model runs carries several caveats. For instance, we have excluded tidal flows, which are found to effectively mediate the heat transfer across the Antarctic continental margin (Stewart et al., 2018) and shape the structures of shelf break fronts (Brink, 2012; Brink & Cherian, 2013). Strong external buoyancy forcing can alter the baroclinic flow field and thus the eddy energy balance over shelves and slopes (Stewart & Thompson, 2016), which may further influence the parameter dependencies of the cross-slope eddy diffusivity. Time-dependent wind forcing can also modulate the slope flow variability via anomalous wind work (Zhai & Greatbatch, 2007) and “eddy memory” effects (Manucharyan et al., 2017). Lastly, eddy-driven cross-slope tracer transfer depends on both eddy buoyancy fluxes (Gent & McWilliams, 1990) and isoneutral eddy mixing (Redi, 1982; Wei & Wang, 2021), but the relative importance of these two processes and their dynamical relationship remain to be investigated for prograde fronts.

Data Availability Statement

The MITgcm code and documentation are available at [http://mitgcm.org](http://mitgcm.org). The data used in this manuscript and the configuration files for the reference MITgcm simulation have been published at [https://doi.org/10.5281/zenodo.7039761](https://doi.org/10.5281/zenodo.7039761).

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### Table 1. List of parameters adopted in the reference model run. *Italics* indicate parameters that are varied between different simulations.

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<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
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<td>$L_y$</td>
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<td>Meridional domain size</td>
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<td>$H_{\text{max}}$</td>
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Table 2. Simulation parameters varied between the model experiments. Bold values indicate parameters that are varied from their reference values. For parameter definitions, the reader is referred to Table 1.

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<th>Experiment Category</th>
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<th>$W_s$(km)</th>
<th>$M_t$(km$^{-1}$)</th>
<th>$Y_t$(km)</th>
<th>$\tau_0$(10$^{-2}$ N/m$^2$)</th>
<th>$\alpha$θ(10$^{-4}$ °C$^{-1}$)</th>
<th>$f_0$(10$^{-4}$ s$^{-1}$)</th>
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Table 3. List of scalings for the eddy buoyancy diffusivity across prograde shelf/slope fronts proposed in previous studies and in this work.

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<th>Prefactor function $F$</th>
<th>Reference diffusivity $K_0$</th>
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<td>$e^{-2</td>
<td>\delta</td>
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<tr>
<td>Stewart and Thompson (2013)</td>
<td>$\delta$</td>
<td>$1 + 0.5\sqrt{(1 -</td>
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<tr>
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<td>$(15 \cdot S + 1)^{-1}$</td>
<td>$u_e L_d$</td>
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<td>Brink and Cherian (2013)</td>
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<td>$(30 \cdot S^2 + 1)^{-1}$</td>
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<td>Brink (2016)</td>
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<td>$u_e^2 / f_0$</td>
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<td>$(7.8 \cdot S^{1.5} + 1)^{-1}$</td>
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<td>$u_e L_E$</td>
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Figures

Figure 1. Illustration of the model bathymetry used in the (a) “Reference” and (b) “4Mt25Yt” simulations (Table 2), superimposed by the snapshots of sea surface potential temperature (color) and the \( \theta = 5^\circ C \) isotherm (blue sheet), and by the magnitude of prograde surface wind stress (black arrows). Time-/zonal-mean eddy kinetic energy as a function of depth and offshore distance for the (c) “Reference” and (d) “4Mt25Yt” simulations, superimposed by selected time-/zonal-mean isopycnals (dashed contours, interval: 2 \( ^\circ C \)) and alongshore velocity profiles (solid contours, interval: 0.2 m/s). The northern sponge layer in the “Reference” run is shadowed with light gray in panel (c), and not shown for the “4Mt25Yt” run (in the latter simulation the sponge layer lies between \( y = 550 \) km and 600 km offshore). In panel (d), both the deepest and shallowest bathymetric contours at each latitude are plotted to illustrate the corrugation of the slope. The latitudes dividing the shelf/slope and slope/open ocean are indicated by dashed black lines in panels (c)–(d).
Figure 2. (a) The diagnosed eddy buoyancy diffusivity $K_\theta$ defined by (5) (black dots) as a function of latitude in the reference simulation. Theoretically estimated diffusivities, constructed using $K_B$ following (11) and $K_{MLT}$ following (22) are indicated by the solid orange curve and dashed orange curve, respectively. Two variants of the GEOMETRIC scaling defined in (15) are also superimposed, with the respective scaling prefactor set as a constant $\alpha = \alpha_0 = 0.07$ (blue curve) and defined as $\alpha = \alpha_0 \cdot F_{GEOM}$ following (18) (green curve). (b) The additive inverse of the slope parameter defined by (6) (black curve) and the slope Burger number $S$ defined by (7) (orange curve) as functions of latitude in the reference simulation. In both panels, latitudes dividing the shelf/slope and slope/open ocean are indicated by dashed black lines, and the northern sponge layer is shadowed with light gray.
Figure 3. Scatter plots of the diagnosed eddy buoyancy diffusivity $K_\theta$ defined by (5) against the theoretically estimated diffusivities, constructed via (a) $K_{S04}$ following (10), (b) $K_{B12}$ following (11), (c) $K_{BC13}$ following (13), and (d) $K_{B16}$ following (14) at each latitude across the analysis regions (cf. Section 3.1) of all “Smooth” simulations (Table 2). Distinct color shades correspond to different magnitudes of the slope Burger number $S$. 
Figure 4. (a) Scatter plot of the diagnosed eddy buoyancy diffusivity $K_\theta$ defined by (5) against the GEOMETRIC scaling $K_{\text{GEOM}}$ defined by (15) with the constant prefactor coefficient optimized as $\alpha = \alpha_0 = 0.07$. Distinct color shades correspond to different magnitudes of the slope Burger number $S$. (b) Illustration of the GEOMETRIC prefactor $\alpha$ calculated following (17), normalized by $\alpha_0$, as a function of the slope Burger number $S$. Diagnostics drawn from the continental slope region (shelf and open ocean regions) are marked in orange (blue). (c) As in (a), but with the $S$-dependent prefactor $\alpha = \alpha_0 \cdot F_{\text{GEOM}}$ defined by (18) incorporated into the GEOMETRIC scaling. In panels (a)–(c), diagnostics are made at each latitude across the analysis regions (cf. Section 3.1) of all “Smooth” simulations (Table 2). (d) As in (c), but with scattered data representing regionally-averaged $K_\theta$ and $K_{\text{GEOM}}$ over continental shelves (circles), continental slopes (dots) and in the open ocean (triangles).
Figure 5. Scatter plots of the diagnosed eddy buoyancy diffusivity $K_e$ defined by (5) against
the mixing length theory-based scalings (a) $K_{MLT}^{Rh}$ defined by (22) and (b) $K_{MLT}^{E}$ defined by (23).
The values of the prefactor $\Gamma$ used in $K_{MLT}^{Rh}$ and $K_{MLT}^{E}$ are 0.17 and 0.08, respectively, to optimize
the logarithmic scaling-diagnosis fit in the open ocean and continental shelf regions. Distinct color
shades correspond to different magnitudes of the slope Burger number $S$. (c) As in (b), but
with the $S$-dependent prefactor $\Gamma \cdot F_{MLT}^{E}$ defined in (24) incorporated into the Eady scale-based
scaling $K_{MLT}^{E}$. In panels (a)–(c), diagnostics are made at each latitude across the analysis regions
(cf. Section 3.1) of all “Smooth” simulations (Table 2). (d) As in (c), but with scattered data
representing regionally-averaged $K_e$ and $K_{MLT}^{E}$ over continental shelves (circles), continental slopes
(dots) and in the open ocean (triangles).
Figure 6. (a) The fastest growth rate of QG linear mode normalized by the Eady (1949) solution $\sigma/(0.31\sigma_E)$ as a function of the slope Burger number $S$ for each water column across the analysis regions (cf. Section 3.1) of all “Smooth” simulations (Table 2). Diagnostics made over continental slopes (in shelf and open ocean regions) are indicated by orange (blue) dots. (b) Probability density functions of the normalized fastest growth rate of QG linear mode for shelf and open ocean regions (blue bars) and for the continental slope regions (orange bars).
Figure 7. (a) Scatter plot of the diagnosed eddy buoyancy diffusivity \( K_b \) defined by (5) against the slope-aware GEOMETRIC scaling \( K_{\text{GEOM}} \) defined by (15) and (18) at each latitude across the analysis regions (cf. Section 3.1) of all “Corrugated” simulations (Table 2). Distinct color shades correspond to different magnitudes of the slope Burger number \( S \). (b) As in (a), but with scattered data representing regionally-averaged \( K_b \) and \( K_{\text{GEOM}} \) over continental shelves (circles), continental slopes (dots) and in the open ocean (triangles). (c) As in (a), but with the theoretically estimated diffusivity constructed via the slope-aware Eady scale-based scaling \( K_{\text{MLT}}^E \) defined by (24). (d) As in (b), but with the theoretically estimated diffusivity constructed via the slope-aware Eady scale-based scaling \( K_{\text{MLT}}^E \) defined by (24).
Appendix A Measure for the Goodness-of-Fit

In this study, we adopt the Nash and Sutcliffe (1970) coefficient of efficiency (see also McCuen et al., 2006; Ritter & Munoz-Carpena, 2013) to quantify the goodness-of-fit between the logarithms of diagnosed and theoretically estimated eddy diffusivities, i.e.,

\[ R^2 = 1 - \frac{\sum_{i=1}^{N_{\text{diag}}} (\log_{10}(K_\theta) - \log_{10}(K_{\text{est}}))^2}{\sum_{i=1}^{N_{\text{diag}}} (\log_{10}(K_\theta) - \overline{\log_{10}(K_\theta)})^2}. \]  

Here \( N_{\text{diag}} \) stands for the total number of diagnostics, \( \overline{\log_{10}(K_\theta)} \) indicates the arithmetic mean operator, \( K_\theta \) and \( K_{\text{est}} \) represent the diagnosed and theoretically estimated eddy diffusivities, respectively. \( R^2 \) defined by (A1) ranges from \(-\infty\) to unity. \( R^2 = 1 \) implies a perfect fit (i.e., \( K_\theta = K_{\text{est}} \)), while a negative \( R^2 \) indicates that the theoretically estimated eddy diffusivity \( K_{\text{est}} \) provides a worse fit to the diagnosed \( K_\theta \) than the arithmetic mean of \( K_\theta \) (upon taking logarithms).

Adopting the logarithms of eddy diffusivities in the calculation of \( R^2 \) emphasizes small-magnitude scaling-diagnosis discrepancies across continental slope regions where baroclinic eddies are most suppressed, and thus highlights the skills of different scalings in capturing the topographic influence on baroclinic eddy fluxes. Calculating \( R^2 \) using the exact values of diffusivities instead of their logarithms yields only quantitative differences in the results reported in this study. Similar logarithm-based measure for the goodness of fit has also been adopted by previous studies when the desired quantities vary across orders of magnitude (Wei & Wang, 2021; Mashayek et al., 2022).

Appendix B Comparisons between different powers of \( S \) in \( F_{\text{GEOM}} \)

Figure B1 presents the scatter plots of the diagnosed eddy buoyancy diffusivities in all “Smooth” simulations against those estimated by the GEOMETRIC scaling, defined following (15) and (18) but with the powers of \( S \) (i.e., \( \mu_2 \) in \( F_{\text{GEOM}} \)) fixed as 1 (Figure B1(a)) and 2 (Figure B1(b)) following Brink (2012), Brink and Cherian (2013), and Brink (2016). The constant factor \( \mu_1 \) in \( F_{\text{GEOM}} \) is re-tuned accordingly to optimize the functional dependence of \( \alpha \) on \( S \) or \( S^2 \) across the continental slope.
Figure B1. (a) Scatter plot of the diagnosed eddy buoyancy diffusivity $K_\theta$ defined by (5) against the GEOMETRIC scaling $K_{\text{GEOM}}$ defined by (15) with the GEOMETRIC prefactor reconstructed as $\alpha = \alpha_0 \cdot (5.1 \cdot S + 1)^{-1}$ at each latitude across the analysis regions of all “Smooth” simulations. (b) As in (a), but with the GEOMETRIC prefactor reconstructed as $\alpha = \alpha_0 \cdot (12.4 \cdot S^2 + 1)^{-1}$. 
References


revealed by the OOI Pioneer Array. *Oceanography*, 31(1).


Strobach, E., Kleine, P., Molod, A., Fahad, A. A., Trayanov, A., Menemenlis, D., & Torres,


