Two-Layer Shallow Water Equations with Complete Coriolis Force and Topography

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Summary. Equations are presented that describe two superposed shallow layers of inviscid fluid flowing over topography in a rotating frame, with a complete treatment of the Coriolis force. Motivated by applications to the Earth's equatorial oceans, these equations offer a physically reasonable alternative to the empirical friction currently used to regularise existing shallow water models at the equator.

1 Introduction

Layered shallow water equations describe the behaviour of several superposed layers of inviscid fluid of different, constant densities flowing over bottom topography, as illustrated in Fig. 1. This structure captures something of the density stratification of the oceans, which makes it a useful idealised setting for studying the interactions between stratification and rotation that govern the large-scale dynamics of the oceans [2, 4, 10].

Shallow water equations may be derived from the three-dimensional Euler equations by averaging the horizontal fluid velocity across each layer. The standard approach neglects the Coriolis terms due to the horizontal components of the Earth's rotation vector, and also the vertical acceleration. These are referred to as the 'traditional' and 'hydrostatic' approximations respectively [10, 12]. Both may be justified in the limit of a vanishingly small ocean depth. However, recent work [4] includes the 'nontraditional' components of the rotation vector in a single-layer shallow water model, and suggests that there may be significant effects associated with these components. This is consistent with the findings of the UK Meteorological Office, who in 1992 abandoned the traditional approximation in their unified model [3, 12].

This paper presents a set of two-layer shallow water equations that incorporate the nontraditional Coriolis terms, but omit the vertical acceleration. They thus correspond to a layered analogue of the quasihydrostatic approximation for a continuously stratified fluid [11, 12]. These equations are particularly relevant for the study of deep ocean currents, such as the Antarctic

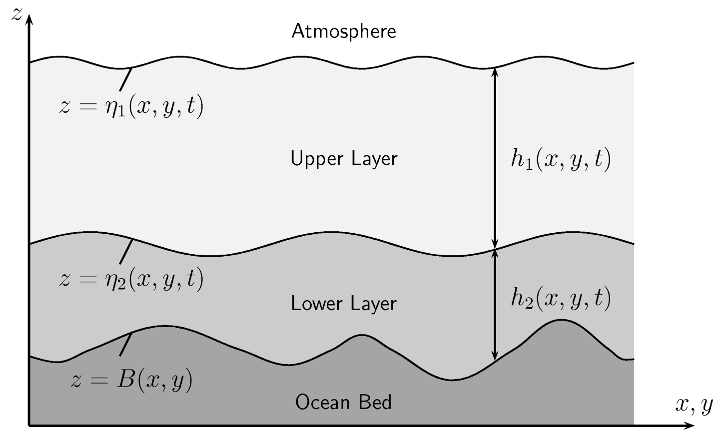

Fig. 1. Structure of the two-layer ocean model

Bottom Water, whose behaviour in equatorial regions is not well described by traditional models [2].

2 Formulation and Derivation

We begin our derivation by writing down the Euler equations for two fluid layers, each of constant density ϱ_i, in a rotating frame,

$$\frac{\partial \tilde{\mathbf{u}}_i}{\partial \tilde{t}} + \left(\tilde{\mathbf{u}}_i \cdot \tilde{\nabla}\right) \tilde{\mathbf{u}}_i + \tilde{w}_i \frac{\partial \tilde{\mathbf{u}}_i}{\partial \tilde{z}} + 2\, \tilde{\Omega}_z \tilde{\mathbf{z}} \times \tilde{\mathbf{u}}_i + 2\, \tilde{\mathbf{\Omega}} \times \hat{\mathbf{z}}\, \tilde{w}_i + \frac{1}{\varrho_i} \tilde{\nabla} \tilde{p}_i = 0, \tag{1a}$$

$$\frac{\partial \tilde{w}_i}{\partial \tilde{t}} + \tilde{\mathbf{u}}_i \cdot \tilde{\nabla} \tilde{w}_i + \tilde{w}_i \frac{\partial \tilde{w}_i}{\partial \tilde{z}} + 2\big(\tilde{v}_i \tilde{\Omega}_x - \tilde{u}_i \tilde{\Omega}_y\big) + \frac{1}{\varrho_i} \frac{\partial \tilde{p}_i}{\partial \tilde{z}} + g = 0, \tag{1b}$$

$$\tilde{\nabla} \cdot \tilde{\mathbf{u}}_i + \frac{\partial \tilde{w}_i}{\partial \tilde{z}} = 0. \tag{1c}$$

Here $i = 1,2$ denotes the upper and lower layers respectively, and the superscript tildes (˜) indicate dimensional variables. The horizontal velocity within each layer is $\tilde{\mathbf{u}}_i = (\tilde{u}_i, \tilde{v}_i)^T$, the vertical velocity is $\tilde{w}_i$, and the pressure is $\tilde{p}_i$. These quantities all depend on $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ and $\tilde{t}$, but $\tilde{\nabla} = (\partial_{\tilde{x}}, \partial_{\tilde{y}})$ is a horizontal derivative. The gravitational acceleration is g, and $\tilde{\mathbf{\Omega}} = (\tilde{\Omega}_x, \tilde{\Omega}_y)^T$ and $\tilde{\Omega}_z$ are the horizontal and vertical components of the rotation vector.

In applying this model we approximate the curved surface of the Earth using a flat plane. The Cartesian coordinates are constructed such that the combination of centrifugal acceleration and gravity acts vertically [10], as represented by the g term in (1b). However, we allow for the spatial variation of the rotation vector with latitude, the so-called β-plane approximation [7, 10]. We thus consider $\tilde{\mathbf{\Omega}} = \tilde{\mathbf{\Omega}}(\tilde{x}, \tilde{y})$ and $\tilde{\Omega}_z = \tilde{\Omega}_z(\tilde{x}, \tilde{y}, \tilde{z})$. In general,

$\tilde{\Omega}_z$ must depend on $\tilde{z}$ to make the three-dimensional rotation vector non-divergent, $\tilde{\nabla} \cdot \tilde{\mathbf{\Omega}} + \partial_z \tilde{\Omega}_z = 0$. This ensures conservation of potential vorticity [7]. Integrating with respect to $\tilde{z}$ yields the following expression for $\tilde{\Omega}_z$, where $\tilde{\Omega}_{z0} = \tilde{\Omega}_z|_{\tilde{z}=0}$,

$$\tilde{\Omega}_z(\tilde{x}, \tilde{y}, \tilde{z}) = \tilde{\Omega}_{z0}(\tilde{x}, \tilde{y}) - (\tilde{\nabla} \cdot \tilde{\mathbf{\Omega}})\tilde{z}. \tag{2}$$

We assume that the upper surface is stress-free ($\tilde{p}_1 = 0$ on $\tilde{z} = \tilde{\eta}_1$) and that the pressure is continuous at the internal surface ($\tilde{p}_1 = \tilde{p}_2$ on $\tilde{z} = \tilde{\eta}_2$). However, we allow for a discontinuous horizontal fluid velocity between the layers, so the kinematic boundary conditions become,

$$\begin{gathered} \tilde{w}_2 = \tilde{\mathbf{u}}_2 \cdot \tilde{\nabla} \tilde{B} \quad \text{on} \quad \tilde{z} = \tilde{B}, \qquad \tilde{w}_1 = \frac{\partial \tilde{\eta}_1}{\partial \tilde{t}} + \tilde{\mathbf{u}}_1 \cdot \tilde{\nabla} \tilde{\eta}_1 \quad \text{on} \quad \tilde{z} = \tilde{\eta}_1, \\ \tilde{w}_2 - \tilde{\mathbf{u}}_2 \cdot \tilde{\nabla} \tilde{\eta}_2 = \frac{\partial \tilde{\eta}_2}{\partial \tilde{t}} = \tilde{w}_1 - \tilde{\mathbf{u}}_1 \cdot \tilde{\nabla} \tilde{\eta}_2 \quad \text{on} \quad \tilde{z} = \tilde{\eta}_2. \end{gathered} \tag{3}$$

We now derive the two-layer shallow water equations by averaging the Euler equations over each layer. We follow a procedure similar to that described in [1, 4], nondimensionalising the govening equations and introducing $\delta = H/L$, the ratio of the vertical to horizontal length scales. We shall take $\delta \ll 1$ below. The resulting set of dimensionless equations is

$$Ro \left(\frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i + w_i \frac{\partial \mathbf{u}_i}{\partial z} \right) + \Omega_z \hat{\mathbf{z}} \times \mathbf{u}_i + \delta \mathbf{\Omega} \times \hat{\mathbf{z}} w_i + \nabla p_i = 0, \tag{4a}$$

$$\delta^2 Ro \left(\frac{\partial w_i}{\partial t} + \mathbf{u}_i \cdot \nabla w_i + w_i \frac{\partial w_i}{\partial z} \right) + \delta (v_i \Omega_x - u_i \Omega_y) + \frac{\partial p_i}{\partial z} + Bu = 0, \tag{4b}$$

$$\nabla \cdot \mathbf{u}_i + \frac{\partial w_i}{\partial z} = 0. \tag{4c}$$

Here $Ro = U/(2\Omega L)$ and $Bu = gH/(2\Omega UL)$ are the Rossby and Burger numbers respectively, which we assume are both O(1). Exploiting $\delta \ll 1$ for shallow layers, we pose asymptotic expansions of $\mathbf{u}_i$, w_i and p_i in the form $\mathbf{u}_i = \mathbf{u}_i^{(0)} + \delta \mathbf{u}_i^{(1)} + \cdots$. Equation (4b) implies that the leading-order pressure $p^{(0)}$ is hydrostatic, so (4a) is satisfied at leading order by a z-independent $\mathbf{u}_i^{(0)}$. We may thus obtain expressions for $w_i^{(0)}$ and $p_i^{(0)} + \delta p_i^{(1)}$ from (4c) and (4b) respectively. The lower layer acquires a contribution to its pressure from the upper layer through the boundary condition at the interface, $p_2^{(0)} + \delta p_2^{(1)} = \varrho_r (p_1^{(0)} + \delta p_1^{(1)})$ on $z = \eta_2$, where $\varrho_r = \varrho_1/\varrho_2$ is the density ratio. Similarly, the vertical velocity in the upper layer acquires a contribution from the vertical velocity in the lower layer, $w_1^{(0)} = w_2^{(0)} - \mathbf{u}_2^{(0)} \cdot \nabla \eta_2 + \mathbf{u}_1^{(0)} \cdot \nabla \eta_1$ on $z = \eta_2$.

Applying the layer averaging formula from [13] to the continuity equation (4c), we derive evolution equations for the layer depths,

$$\frac{\partial h_i}{\partial t} + \nabla \cdot (h_i \overline{\mathbf{u}}_i) = 0, \tag{5}$$

where an overbar ($\bar{\ }$) denotes a layer average,

$$\overline{\mathbf{u}}_1 = \frac{1}{h_1}\int_{\eta_2}^{\eta_1} \mathbf{u}_1 \mathrm{d}z, \quad \overline{\mathbf{u}}_2 = \frac{1}{h_2}\int_{B}^{\eta_2} \mathbf{u}_2 \mathrm{d}z. \tag{6}$$

The depths of the two layers are $h_1 = \eta_1 - \eta_2$ and $h_2 = \eta_2 - B$.

Averaging (4a) over each layer, as described in [1, 4], we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\left(h_i \overline{\mathbf{u}}_i\right) + \nabla\cdot\left(h_i \overline{\mathbf{u}_i \mathbf{u}_i}\right) + h_i \hat{\mathbf{z}} \times \overline{\Omega_z \mathbf{u}_i} \\ + \delta \boldsymbol{\Omega} \times \hat{\mathbf{z}} h_i \overline{w_i^{(0)}} + h_i \overline{\nabla\left(p_i^{(0)} + \delta p_i^{(1)}\right)} = \mathrm{O}(\delta^2). \end{aligned} \tag{7}$$

The average pressure gradient may be computed from $p_i^{(0)} + \delta p_i^{(1)}$, as found above. To complete the derivation, we note that we may factorise the averages of products of quantities that are z-independent to leading order [1, 9]. Since $\mathbf{u}_i = \mathbf{u}_i^{(0)} + \mathrm{O}(\delta)$, $\overline{\mathbf{u}_i \mathbf{u}_i} = \overline{\mathbf{u}}_i \overline{\mathbf{u}}_i + \mathrm{O}(\delta^2)$. Similarly, $\overline{\Omega_z \mathbf{u}_i} = \overline{\Omega_z}\,\overline{\mathbf{u}}_i + \mathrm{O}(\delta^2)$, we may evaluate $\overline{\Omega_z}$ using (2), and $\mathbf{u}_i^{(0)} = \overline{\mathbf{u}}_i + \mathrm{O}(\delta)$. Neglecting terms of $\mathrm{O}(\delta^2)$ and above, we obtain the following averaged momentum equations,

$$\begin{aligned} Ro\left(\frac{\partial \mathbf{u}_1}{\partial t} + (\mathbf{u}_1\cdot\nabla)\mathbf{u}_1\right) + \left[\Omega_{z0} - \delta\nabla\cdot\left(\left(B + h_2 + \tfrac{1}{2}h_1\right)\boldsymbol{\Omega}\right)\right]\hat{\mathbf{z}}\times\mathbf{u}_1 \\ + \nabla\left[Bu\left(B + h_2 + h_1\right) + \tfrac{1}{2}\delta h_1 (v_1\Omega_x - u_1\Omega_y)\right] \\ - \delta\,\boldsymbol{\Omega}\times\hat{\mathbf{z}}\,\nabla\cdot\left(h_2\mathbf{u}_2 + \tfrac{1}{2}h_1\mathbf{u}_1\right) = 0, \end{aligned} \tag{8a}$$

$$\begin{aligned} Ro\left(\frac{\partial \mathbf{u}_2}{\partial t} + (\mathbf{u}_2\cdot\nabla)\mathbf{u}_2\right) + \left[\Omega_{z0} - \delta\nabla\cdot\left(\left(B + \tfrac{1}{2}h_2\right)\boldsymbol{\Omega}\right)\right]\hat{\mathbf{z}}\times\mathbf{u}_2 \\ + \nabla\Big[Bu\left(B + h_2 + \varrho_r h_1\right) + \tfrac{1}{2}\delta h_2 (v_2\Omega_x - u_2\Omega_y) \\ + \delta\varrho_r h_1 \left(v_1\Omega_x - u_1\Omega_y\right)\Big] - \delta\,\boldsymbol{\Omega}\times\hat{\mathbf{z}}\,\nabla\cdot\left(\tfrac{1}{2}h_2\mathbf{u}_2\right) = 0. \end{aligned} \tag{8b}$$

Here we have dropped the overbars on averaged velocities. We thus obtain the traditional two layer shallow water equations [10] plus several additional terms proportional to Ω_x and Ω_y.

3 Conservation Properties

The 'nontraditional' two-layer shallow water equations inherit the conservation laws of the full three-dimensional equations. In particular, there are two materially conserved potential vorticities, $\partial_t q_i + \mathbf{u}_i \cdot \nabla q_i = 0$ for $i = 1,2$, with

$$q_i = \frac{1}{h_i}\left\{\left[\Omega_{z0} - \delta\nabla\cdot\left(\left(\eta_i - \frac{h_i}{2}\right)\boldsymbol{\Omega}\right)\right] + Ro\left(\frac{\partial v_i}{\partial x} - \frac{\partial u_i}{\partial y}\right)\right\}. \tag{9}$$

These q_i differ by terms proportional to Ω_x and Ω_y from the traditional potential vorticities given in [10]. This modification may provide useful insight into the dynamics of cross-equatorial ocean currents. The contributions from Ω_{z0} change sign at the equator, which severely constrains the ability of fluid parcels to cross the equator [8]. This constraint may be at least partly alleviated by the additional contribution to the q_i from the interaction of topography and nontraditional rotation.

Conservation laws for the energy and momentum of the fluid may also be obtained. The energy density is unchanged by rotation, whilst the energy flux and momentum density acquire additional terms containing nontraditional components of the rotation vector.

4 Linear Plane Waves

Some important properties of the extended shallow water equations may be highlighted by considering linear plane wave solutions. Taking the usual GFD axes (y pointing north, x pointing east) and a non-traditional f-plane approximation [10], such that $\Omega_x = 0$ and Ω_y and Ω_z are constants, we linearise (8a), (8b) and (5) by assuming that the dependent variables are small perturbations to a state of rest: $\mathbf{u}_1 = \mathbf{u}_1'$, $\mathbf{u}_2 = \mathbf{u}_2'$, $h_1 = H_1 + h_1'$ and $h_2 = H_2 + h_2'$. By neglecting products of these variables and seeking solutions of the form $\exp\left(\mathrm{i}(kx + ly - \omega t)\right)$, we obtain a dispersion relation for the waves.

This dispersion relation is plotted in Fig. 2. We have taken the layers to be of equal mean depth ($H_1 = H_2$), and the aspect ratio to be $\delta = 0.2$, a little larger than typical for long internal waves ($0.02 \lesssim \delta \lesssim 0.14$). Our density ratio is $\varrho_r = 0.9$. The realistic value $\varrho_r = 0.98$ makes it impossible to show the two wave branches on the same plot. We have also set $Ro = Bu = 1$ for the purpose of illustration. The waves with higher frequency propagate on the internal and upper surfaces simultaneously. The lower frequency waves propagate primarily on the interface between the layers, with the upper surface remaining approximately flat. The nontraditional Coriolis effects cause a distinct shift in the frequencies, creating a left/right asymmetry. More importantly, nontraditional effects create a range of waves with frequencies below the inertial frequency, the smallest allowable frequency under the traditional approximation. These so-called subinertial waves have been observed in previous studies of nontraditional Coriolis effects in continuously stratified fluids [5, 6], and provide an additional source of energy for mixing in the deep ocean.

5 Conclusions

We have derived two-layer shallow water equations that include additional terms arising from the nontraditional components of the Coriolis force. They may be shown to retain the expected conservation laws for energy, momentum, and potential vorticity. We have illustrated some deviations from traditional behaviour, such as the existence of subinertial waves, caused by the additional

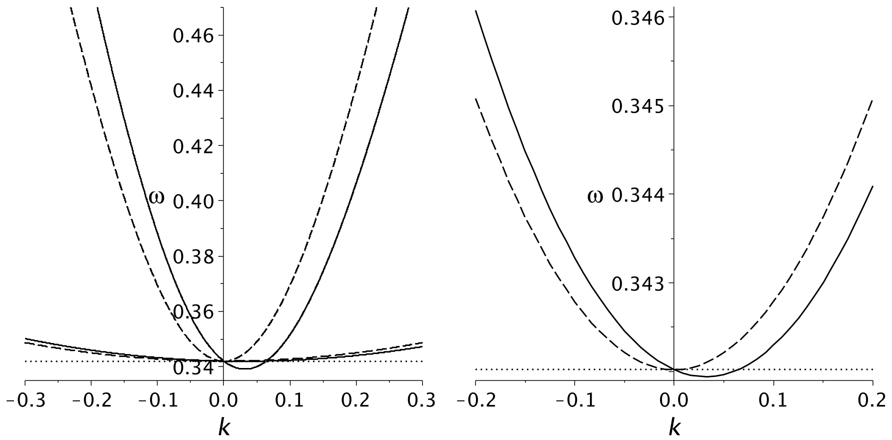

Fig. 2. Dispersion relation for waves propagating zonally at a latitude of 10 ° North in the traditional (dashed line) and nontraditional (solid line) two-layer shallow water equations. Left: all wave modes. Right: internal wave modes. Notice the band of internal waves with frequencies below the inertial frequency (dotted line).

part of the Coriolis force. These equations will serve as a useful prototype for investigating the dynamics of cross-equatorial ocean currents over topography.

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