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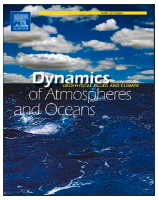

Comments on “horizontal gravity disturbance vector in atmospheric dynamics” by Peter C. Chu

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ABSTRACT

In a recent paper [Chu (2023; Chu23)], the author formulated the equations governing atmospheric motion in a spheroidal coordinate system. Since the mass distribution of the Earth is not exactly spheroidal, the true gravity is not vertical in that coordinate system. Chu23 compared the magnitude of the static horizontal component of gravity in that system to those of the dynamically active forces and concluded that the horizontal components of gravity should not be neglected. In recent papers by the authors [Chang and Wolfe (2022; CW22) and Stewart and McWilliams (2022; CW22)], we explained that the actual interpretation of the approximation made in atmospheric and oceanic modeling is not neglecting the horizontal component of the true gravity, but is a geometrical approximation, approximating nearly spheroidal geopotential surfaces with bumps on which the true gravity is vertical by exactly spheroidal surfaces. We showed that under such an interpretation, the errors due to the geometrical approximation are small. Chu23 claimed that CW22 and SM22 erroneously neglected the gravity perturbations in their analyses. Here, we explain further the differences between these approaches, in the process showing that the criticisms of Chu23 on CW22 and SM22 are invalid, further supporting our conclusion that the horizontal component of the true gravity is not relevant in ocean and atmospheric dynamics. Physically, the reason why horizontal gravity is irrelevant in the coordinate system used by Chu23 is that it is balanced by a static horizontal pressure gradient force.

The rotation of the Earth produces a centrifugal force that distorts the Earth’s mass distribution from spherical to nearly spheroidal with small spatial inhomogeneities (e.g., Todhunter, 1873; Förste et al., 2014; Staniforth, 2022). The geopotential—the sum of the gravitational potential produced by this mass distribution and the centrifugal potential—is also nearly spheroidal. If Earth’s mass distribution were exactly spheroidal, the geopotential would also be exactly spheroidal, and net gravity due to this hypothetical geopotential would be perpendicular to spheroidal surfaces—this is the $\mathbf{g}_{\mathrm{eff}}$ defined by Chu (2023; hereafter Chu23). However, the Earth’s mass distribution is not exactly spheroidal, and the (slightly) uneven mass distribution gives rise to a perturbation field $\delta\mathbf{g}$. The true (or total) gravity $\mathbf{g}$ is the sum of $\mathbf{g}_{\mathrm{eff}}$ and $\delta\mathbf{g}$. Chang and Wolfe (2022; hereafter CW22) and Stewart and McWilliams (2022; hereafter SM22) pointed out that atmospheric and oceanic scientists express the equations of motion in coordinate form by defining the “vertical” direction in the coordinate system to be opposite to $\mathbf{g}$, effectively using a geopotential coordinate (see, e.g., Gill, 1982 – see Appendix A). Importantly, in this coordinate system, the true gravity, $\mathbf{g} = \mathbf{g}_{\mathrm{eff}} + \delta\mathbf{g}$, is *exactly* vertical—with no horizontal components.

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Furthermore, in this coordinate system, “horizontal” geopotential surfaces are not exactly spheroidal but are nearly spheroids with some bumps due to the inhomogeneities of the Earth’s mass distribution. For mathematical simplicity, atmospheric and oceanic scientists approximate these geopotential coordinate surfaces *geometrically* as exact spheroids; that is, they use a coordinate system in which true gravity is exactly aligned with the vertical coordinate r and approximate the shapes of the iso-surfaces of r as spheroids. For clarity, we will henceforth refer to this approximation as the *spheroidal geopotential approximation*.

Let us contrast the approach of CW22 and SM22 with that of Chu23: Chu23 started with an exact spheroidal coordinate system in which coordinate surfaces are perpendicular to $\mathbf{g}_{\mathrm{eff}}$ instead of $\mathbf{g}$. In this coordinate system, the horizontal components of gravity are not zero, due to $\delta\mathbf{g}$. Chu23 incorrectly claims that the standard formulation of atmospheric and oceanic models is similarly posed in an exact spheroidal coordinate system and neglects $\delta\mathbf{g}$ in favor of $\mathbf{g}_{\mathrm{eff}}$; we will henceforth refer to this as the *absolute spheroidal approximation*, to differentiate from the *spheroidal geopotential approximation* discussed in the preceding paragraph. The analysis in Chu23 evaluates the error in the absolute spheroidal approximation by comparing the magnitude of the horizontal component of $\delta\mathbf{g}$ to the magnitude of dynamically active horizontal forces, such as the Coriolis force. For example, the non-dimensional C number in Chu23 is the ratio of the horizontal component of $\delta\mathbf{g}$ to that of the Coriolis force. Chu23 found that the average magnitude of the C number is slightly less than 1 and therefore not negligible, and thus claimed that the “neglect” of $\delta\mathbf{g}$ is not warranted. However, as noted by CW22 and SM22, this analysis only quantifies the error introduced by making the *absolute spheroidal approximation*; that is, neglecting the horizontal component of gravity in an absolute spheroidal coordinate system. It does not quantify the error in the community-standard *spheroidal geopotential approximation* described in the preceding paragraph; that is, in adopting geopotential coordinates and then approximating the shapes of the geopotentials as spheroids. As shown by CW22, the metric errors introduced in the calculus of the *spheroidal geopotential approximation* are small, reaffirming the long-standing practice of using this coordinate system for atmospheric and oceanic modeling (Gill, 1982; Staniforth, 2022). Based on these and similar analyses, CW22 and SM22 concluded that the horizontal components of the true gravity are not relevant to ocean (and atmospheric) dynamics because these horizontal components vanish when the coordinate system is interpreted correctly. Please refer to CW22 and SM22 for details of the analyses, which will not be reproduced here.

As noted by CW22, the fundamental issue underlying the analysis of Chu23 [and its predecessor, Chu (2021)], can be further illustrated with reference to another community-standard approximation: the approximation of spheroidal geopotential surfaces as spherical, which we will refer to as the *spherical geopotential approximation* (see Staniforth, 2022 for a detailed treatment). This is analogous to the *spheroidal geopotential approximation* described above: the vertical coordinate is aligned with geopotentials, and then those geopotentials are approximated as spheres instead of spheroids. This approximation is also adopted by Chu23, stating that the errors of such an approximation are small (last paragraph in section 2.2 of Chu23). It is inconsistent of Chu23 to apply this *spherical geopotential approximation* while insisting that the *spheroidal geopotential approximation* cannot be applied to the smaller variations in the geopotential field due to the Earth’s uneven mass distribution. If we were to apply Chu23′s analysis to the *spherical geopotential approximation*, we would be forced to interpret this approximation not as an approximation of the geometry, but rather as neglecting the horizontal component of $\mathbf{g}_{\mathrm{eff}}$ in an exact spherical coordinate system. If we proceeded with Chu23′s analysis and compared the magnitude of the “neglected horizontal” component of $\mathbf{g}_{\mathrm{eff}}$ in an exact spherical coordinate system to the Coriolis force (equivalent to the C number of Chu23), we would find that $C > 10$. Using Chu’s reasoning, this would imply that the *spherical geopotential approximation* is grossly inaccurate. On the contrary, this apparent paradox is resolved in the community-standard treatment of the *spherical geopotential approximation* (see Staniforth, 2022) by redefining the vertical direction to be opposite to $\mathbf{g}_{\mathrm{eff}}$, such that the horizontal component of $\mathbf{g}_{\mathrm{eff}}$ becomes exactly zero. The approximation then becomes an approximation of the geometry (i.e., approximating spheroids as spheres) rather than the neglect of the horizontal component of $\mathbf{g}_{\mathrm{eff}}$, resulting in errors that are small (e.g., Bénard, 2015). This implies that comparing the magnitude of dynamically active forces to the “neglected” horizontal component of $\mathbf{g}_{\mathrm{eff}}$ in a spherical coordinate system is not relevant for assessing the quality of the *spherical geopotential approximation.* The *spheroidal geopotential approximation* of CW22 and SM22 [as described in Gill (1982)] extends this *spherical geopotential approximation* to approximating *nearly* spheroidal geopotential surfaces (with perturbations due to uneven gravity) as spheroidal surfaces. The errors are again geometrical and are estimated to be small by CW22. These analyses demonstrate that the horizontal components of gravity are not relevant to ocean and atmospheric dynamics, and that gravity should be considered to be vertical.

Physically, as pointed out by CW22 and SM22, the reason why the horizontal components of gravity in a spheroidal (or spherical) coordinate system are not dynamically relevant is that in a fluid, static forces are largely balanced by a static pressure gradient force. The presence of horizontal gravity in the equations of motion will drive a static horizontal pressure gradient force that largely cancels this component of gravity. Failure to account for this cancelation is also the fundamental flaw of Chu (2021), in which the author assumed that the horizontal components of gravity will drive Ekman transport instead of being largely balanced by a static horizontal pressure gradient force in spheroidal coordinates (see equations 17–20 of Chu, 2021). In such a coordinate system, horizontal dynamical balance appears as a residual of the dominant horizontal static balance. The use of geopotential coordinates, in which gravity is exactly vertical, eliminates this horizontal static balance and highlights the dynamical balance.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. : Discussion of gravity perturbations in Gill (1982)

Chu23 claimed that the textbooks cited by us only provided information about $\mathbf{g}_{\mathrm{eff}}$, implying that they did not mention the gravity perturbations represented by $\delta\mathbf{g}$. This is not true. Let us quote from Gill (1982, from p.91):

> “Because the gravitational force is so dominant in the equations of motion, great care is required [see N. A. Phillips (1973)] in defining a suitable coordinate system. … It is therefore preferable to use geopotential surfaces rather than spheres in defining the coordinate system.
>
> The shape of geopotential surfaces is known quite well from satellite data. As a first approximation, the sea-level geopotential surface is an oblate spheroid … Departures from this ellipsoid of best fit are relatively small and can be shown on a map [e.g. Fig. 3 in the work by Lerch et al. (1979)]. The largest departure is a depression of ‘depth’ 100 m just to the south of India …”

Note that Fig. 3a of Chu23 showed exactly what was described by Gill (1982). Hence it is clear that Gill (1982) discussed this gravity perturbation in the textbook, and that Chu23 was erroneous in stating that the textbooks only provided information on $\mathbf{g}_{\mathrm{eff}}$ and not $\delta\mathbf{g}$.

Later, on the same page, Gill (1982) wrote:

> “The coordinate system to be used is a slight distortion from the oblate spheroidal set … The difference between the expressions for the various terms in the equations is so small that it can be ignored, with the exception of the fact that in the system to be used gravity is exactly perpendicular to the surfaces $r =$ constant.”

Gill (1982) did not provide an estimate of the errors but stated that they were small. CW22 provided a quantitative estimate of this error and showed that the errors were indeed small as suggested by Gill (1982).

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