The Complex Physics of Climate Change: Nonlinearity and Stochasticity

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Please visit these sites for more info.
http://www.atmos.ucla.edu/tcd/
http://www.environnement.ens.fr/
Motivation

- The **climate system** is highly **nonlinear and complex**.
- Its **major components** — the atmosphere, oceans, ice sheets — **evolve on many space and time scales**.
- Its **predictive understanding** has to rely on the system’s physical, chemical and biological **modeling**, but also on the **mathematical analysis** of the models thus obtained.
- The **hierarchical modeling** approach allows one to give proper weight to the **understanding provided by the models vs. their realism**, respectively.
- Back-and-forth between “toy” (conceptual) and **detailed models** (“realistic”) and between **models and data**.
- Such an approach facilitates the evaluation of **forecasts (prognostications?)** based on these models.
Outline

• The IPCC process: results and further questions
• Natural climate variability as a source of uncertainties
  – sensitivity to initial data → error growth
  – sensitivity to model formulation → see below!
• Uncertainties and how to fix them
  – structural in/stability
  – random dynamical systems (RDS)
• Two or more illustrative examples
  – Arnol’d tongues and a “French garden”
  – the Lorenz system
  – an ENSO “toy” model
• Linear response theory and climate sensitivity
• Conclusions, work in progress and references
Greenhouse gases (GHGs) go up, temperatures go up:

It’s gotta do with us, at least a bit, doesn’t it?

Wikicommmons, from Hansen et al. (PNAS, 2006); see also http://data.giss.nasa.gov/gistemp/graphs/
Unfortunately, things aren’t all that easy!

What to do?

Try to achieve better interpretation of, and agreement between, models …


Natural variability introduces additional complexity into the anthropogenic climate change problem

The most common interpretation of observations and GCM simulations of climate change is still in terms of a scalar, linear Ordinary Differential Equation (ODE)

\[
\frac{dT}{dt} = -kt + Q
\]

\[
k = \sum k_i - \text{feedbacks (+ve and -ve)}
\]

\[
Q = \sum Q_j - \text{sources & sinks}
\]

\[
Q_j = Q_j(t)
\]

Linear response to CO₂ vs. observed change in T

Hence, we need to consider instead a system of nonlinear Partial Differential Equations (PDEs), with parameters and multiplicative, as well as additive forcing

\[
\frac{dX}{dt} = N(X, t, \mu, \beta)
\]
Global warming and its socio-economic impacts

Temperatures rise:
- What about impacts?
- How to adapt?

The answer, my friend, is blowing in the wind, i.e., it depends on the accuracy and reliability of the forecast …

Source: IPCC (2007), AR4, WGI, SPM
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<th>Deterministic predictions</th>
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Ensemble forecast of Lothar (surface pressure)
Start date 24 December 1999 : Forecast time T+42 hours

Courtesy Tim Palmer, 2009
Table SPM.2. Recent trends, assessment of human influence on the trend and projections for extreme weather events for which there is an observed late-20th century trend. (Tables 3.7, 3.8, 9.4; Sections 3.8, 5.5, 9.7, 11.2–11.9)

<table>
<thead>
<tr>
<th>Phenomenon(^a) and direction of trend</th>
<th>Likelihood that trend occurred in late 20th century (typically post 1960)</th>
<th>Likelihood of a human contribution to observed trend(^b)</th>
<th>Likelihood of future trends based on projections for 21st century using SRES scenarios</th>
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<tbody>
<tr>
<td>Warmer and fewer cold days and nights over most land areas</td>
<td>Very likely(^c)</td>
<td>Likely(^d)</td>
<td>Virtually certain(^d)</td>
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<tr>
<td>Warmer and more frequent hot days and nights over most land areas</td>
<td>Very likely(^a)</td>
<td>Likely (nights)(^d)</td>
<td>Virtually certain(^d)</td>
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<tr>
<td>Warm spells/heat waves. Frequency increases over most land areas</td>
<td>Likely</td>
<td>More likely than not(^f)</td>
<td>Very likely</td>
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<tr>
<td>Heavy precipitation events. Frequency (or proportion of total rainfall from heavy falls) increases over most areas</td>
<td>Likely</td>
<td>More likely than not(^f)</td>
<td>Very likely</td>
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<td>Area affected by droughts increases</td>
<td>Likely in many regions since 1970s</td>
<td>More likely than not</td>
<td>Likely</td>
</tr>
<tr>
<td>Intense tropical cyclone activity increases</td>
<td>Likely in some regions since 1970</td>
<td>More likely than not(^f)</td>
<td>Likely</td>
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<tr>
<td>Increased incidence of extreme high sea level (excludes tsunamis)(^g)</td>
<td>Likely</td>
<td>More likely than not(^h)</td>
<td>Likely (^i)</td>
</tr>
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How important are different sources of uncertainty?

- Varies, but typically no single source dominates.
Consider the scalar, linear ordinary differential equation (ODE)
\[ \dot{x} = -\alpha x + \sigma t, \quad \alpha > 0, \quad \sigma > 0. \]

The autonomous part of this ODE, \( \dot{x} = -\alpha x \), is dissipative and all solutions \( x(t; x_0) = x(t; x(0) = x_0) \) converge to 0 as \( t \to +\infty \).

What about the non-autonomous, forced ODE? As the energy being put into the system by the forcing is dissipated, we expect things to change in time. In fact, if we “pull back” far enough, replace \( x(t; x_0) \) by \( x(s, t; x_0) = x(s, t; x(s) = x_0) \),

and let \( s \to -\infty \),

we get the pullback attractor \( a = a(t) \) in the figure.
We've just shown that:

\[ |x(t, s; x_0) - a(t)| \rightarrow 0 \text{ as } s \rightarrow -\infty \]

for every \( t \) fixed, and for all initial data \( x_0 \), with \( a(t) = \frac{\sigma}{\alpha} (t - 1/\alpha) \).

We've just encountered the concept of pullback attraction; here \( \{a(t)\} \) is the pullback attractor of the system (1).

What does it mean physically?

The pullback attractor provides a way to assess an asymptotic regime at time \( t \) — the time at which we observe the system — for a system starting to evolve from the remote past \( s, s << t \).

This asymptotic regime evolves with time: it is a dynamical object.

Dissipation now leads to a dynamic object rather than to a static one, like the strange attractor of an autonomous system.
Remarks

We’ve just shown that:

$$|x(t, s; x_0) - a(t)| \xrightarrow{s \to -\infty} 0; \text{ for every } t \text{ fixed,}$$

and for all initial data $x_0$, with $a(t) = \frac{\sigma}{\alpha} (t - 1/\alpha)$.

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The pullback attractor provides a way to assess an asymptotic regime at time $t$ — the time at which we observe the system — for a system starting to evolve from the remote past $s$, $s << t$.

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Dissipation now leads to a dynamic object rather than to a static one, like the strange attractor of an autonomous system.
Random Dynamical Systems - RDS theory

This theory is a combination of measure (probability) theory and dynamical systems, treated systematically by the "Bremen group" (L. Arnold, 1998). It allows one to treat stochastic differential equations (SDEs), and more general systems driven by some "noise," as flows.

Setting:

(i) A phase space \(X\). **Example:** \(\mathbb{R}^n\).

(ii) A probability space \((\Omega, \mathcal{F}, \mathbb{P})\). **Example:** The Wiener space \(\Omega = C_0(\mathbb{R}; \mathbb{R}^n)\) with Wiener measure \(\mathbb{P} = \gamma\).

(iii) A model of the noise \(\theta(t) : \Omega \rightarrow \Omega\) that preserves the measure \(\mathbb{P}\), i.e. \(\theta(t)\mathbb{P} = \mathbb{P}\); \(\theta\) is called the driving system. **Example:** \(W(t, \theta(s)\omega) = W(t + s, \omega) - W(s, \omega)\); it starts the noise at \(s\) instead of \(t = 0\).

(iv) A mapping \(\varphi : \mathbb{R} \times \Omega \times X \rightarrow X\) with the cocycle property. **Example:** The solution of an SDE.
A random attractor $A(\omega)$ is both *invariant* and “pullback" *attracting*:

(a) **Invariant**: $\varphi(t, \omega)A(\omega) = A(\theta(t)\omega)$.

(b) **Attracting**: $\forall B \subset X, \lim_{t \to \infty} \text{dist}(\varphi(t, \theta(-t)\omega)B, A(\omega)) = 0$ a.s.
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A snapshot of the RA, $\mathcal{A}(\omega)$, computed at a fixed time $t$ and for the same realization $\omega$; it is made up of points transported by the stochastic flow, from the remote past $t - T$, $T >> 1$.

We use multiplicative noise in the deterministic Lorenz model, with the classical parameter values $b = 8/3$, $\sigma = 10$, and $r = 28$.

Even computed pathwise, this object supports meaningful statistics.
We can compute the probability measure on the R.A. at some fixed time \( t \). We show a “projection”, \( \int \mu_{\omega}(x, y, z)dy \), with multiplicative noise:
\[
dx_i = \text{Lorenz}(x_1, x_2, x_3)dt + \alpha x_i dW_t; \ i \in \{1, 2, 3\}.
\]

10 million of initial points have been used for this picture!
Still 1 Billion I.D., and $\alpha = 0.3$. 
Sample measure supported by the R.A.

Still 1 Billion I.D., and $\alpha = 0.5$. Another one?
Sample measures evolve with time.

- Recall that these sample measures are the **frozen statistics** at a time $t$ for a realization $\omega$.

- How do these **frozen statistics** evolve with time?

- **Action!**
A day in the life of the Lorenz (1963) model’s random attractor, or LORA for short; see SI in Chekroun, Simonnet & Ghil (2011, *Physica D*)
Applications to a nonlinear stochastic El Niño model

Chekroun, Simonnet and Ghil, 2011

Timmerman & Jin (Geophys. Res. Lett., 2002) have derived the following low-order, tropical-atmosphere–ocean model. The model has three variables: thermocline depth anomaly \( h \), and SSTs \( T_1 \) and \( T_2 \) in the western and eastern basin.

\[
\begin{align*}
\dot{T}_1 &= -\alpha(T_1 - T_r) - \frac{2\varepsilon u}{L}(T_2 - T_1), \\
\dot{T}_2 &= -\alpha(T_2 - T_r) - \frac{w}{Hm}(T_2 - T_{sub}), \\
\dot{h} &= r(-h - bL\tau/2).
\end{align*}
\]

The related diagnostic equations are:

\[
\begin{align*}
T_{sub} &= T_r - \frac{T_r - T_{r0}}{2}[1 - \tanh(H + h_2 - z_0)/h^*] \\
\tau &= \frac{a}{\beta}(T_1 - T_2)[\xi_t - 1].
\end{align*}
\]

- \( \tau \): the wind stress anomalies, \( w = -\beta \tau/H_m \): the equatorial upwelling.
- \( u = \beta L \tau/2 \): the zonal advection, \( T_{sub} \): the subsurface temperature.

Wind stress bursts are modeled as white noise \( \xi_t \) of variance \( \sigma \), and \( \varepsilon \) measures the strength of the zonal advection.
Random attractors: the frozen statistics

Random Shil’nikov horseshoes

Horseshoes can be noise-excited,
left: a weakly-perturbed limit cycle, right: the same with larger noise.

Golden: most frequently-visited areas; white ’plus’ sign: most visited.

\[ \sigma = 0.005 \quad \text{and} \quad \sigma = 0.05 \]
An episode in the random’s attractor life
The IPCC process: results and further questions.

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Climate and Its Sensitivity

Let’s say CO$_2$ doubles:

How will “climate” change?

1. Climate is in **stable equilibrium** (fixed point); if so, mean temperature will just shift gradually to its new equilibrium value.

2. Climate is **purely periodic**; if so, **mean temperature** will (maybe) shift gradually to its new equilibrium value. But how will the **period, amplitude and phase** of the limit cycle change?

3. And how about some “real stuff” now: chaotic + random?

Ghil (Encycl. Global Environmental Change, 2002)
Physically closed system, modeled mathematically as autonomous system: neither deterministic (anthropogenic) nor random (natural) forcing.

The attractor is strange, but still fixed in time ~ “irrational” number.

Climate sensitivity ~ change in the average value (first moment) of the coordinates \((x, y, z)\) as a parameter \(\lambda\) changes.
Physically open system, modeled mathematically as non-autonomous system: allows for deterministic (anthropogenic) as well as random (natural) forcing.

The attractor is “pullback” and evolves in time ~ “imaginary” or “complex” number.

Climate sensitivity ~ change in the statistical properties (first and higher-order moments) of the attractor as one or more parameters ($\lambda$, $\mu$, ...) change.

Ghil (Encyclopedia of Atmospheric Sciences, 2nd ed., 2012)
Parameter dependence – I

It can be smooth or it can be rough:
Niño-3 SSTs from intermediate coupled model for ENSO (Jin, Neelin & Ghil, 1994, 1996)

Skewness & kurtosis of the SSTs: time series of 4000 years,

\[ \Delta \delta = 3 \cdot 10^{-4} \]

\( \delta = 0.9557 \)

M. Chekroun & D. Kondrashov (work in progress)
When it is smooth, one can optimize a GCM's single-parameter dependence.
Multi-objective algorithms avoid arbitrary weighting of criteria in a unique cost function:

Optimization algorithms that are $O(N)$ and $O(N^2)$, rather than $O(S^N)$, where $N$ is the number of parameters and $S$ is the sampling density.

ICTP AGCM (Neelin, Bracco, Luo, McWilliams & Meyerson, *PNAS*, 2010)
Sample measures for an NDDE model of ENSO

The Galanti-Tziperman (GT) model (JAS, 1999)

\[
\frac{dT}{dt} = -\epsilon_T T(t) - M_0(T(t) - T_{sub}(h(t))),
\]

\[
h(t) = M_1 e^{-\epsilon_m(\tau_1 + \tau_2)}h(t - \tau_1 - \tau_2)
    - M_2 \tau_1 e^{-\epsilon_m(\tau_1/2 + \tau_2)}\mu(t - \tau_2 - \tau_1/2)T(t - \tau_2 - \tau_1/2)
    + M_3 \tau_2 e^{-\epsilon_m \tau_2/2} \mu(t - \tau_2/2)T(t - \tau_2/2).
\]

Neutral delay-differential equation (NDDE), derived from Cane-Zebiak and Jin-Neelin models for ENSO: \(T\) is East-basin SST and \(h\) is thermocline depth.

Seasonal forcing given by \(\mu(t) = 1 + \epsilon \cos(\omega t + \phi)\). The pullback attractor and its invariant measures were computed.

Figure shows the changes in the mean, 2\(^{nd}\) & 4\(^{th}\) moment of \(h(t)\), along with the Wasserstein distance \(d_W\), for changes of 0–5\% in the delay parameter \(\tau_{K,0}\).

Note intervals of both smooth & rough dependence!
The time-dependent pullback attractor of the GT model supports an invariant measure $\nu = \nu(t)$, whose density is plotted in 3-D perspective.

The plot is in delay coordinates $h(t+1)$ vs. $h(t)$ and the density is highly concentrated along 1-D filaments and, furthermore, exhibits sharp, near-0-D peaks on these filaments.

The Wasserstein distance $d_W$ between one such configuration, at given parameter values, and another one, at a different set of values, is proportional to the work needed to move the total probability mass from one configuration to the other.

Climate sensitivity $\gamma$ can be defined as $\gamma = \partial d_W / \partial \tau$.
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Concluding remarks, I – RDS and RAs

Summary
• A change of paradigm for open, non-autonomous systems
• Random attractors are (i) spectacular, (ii) useful, and (iii) just starting to be explored for climate applications.

Work in progress
• Study the effect of specific stochastic parametrizations on model robustness.
• Applications to intermediate models and GCMs.
• Implications for climate sensitivity.
• Implications for predictability?
Some general references


Reserve slides
Atmospheric CO₂ at Mauna Loa Observatory

Scripps Institution of Oceanography
NOAA Earth System Research Laboratory

PARTS PER MILLION

YEAR


320  340  360  380
**GHGs rise!**

It’s gotta do with us, at least a bit, ain’t it?

But just how much?

---

*IPCC (2007)*

---

**Radiative Forcing Components**

<table>
<thead>
<tr>
<th>RF Terms</th>
<th>RF values (W m⁻²)</th>
<th>Spatial scale</th>
<th>LOSU</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long-lived greenhouse gases</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CO₂</td>
<td>1.66 [1.49 to 1.83]</td>
<td>Global</td>
<td>High</td>
</tr>
<tr>
<td>N₂O</td>
<td>0.48 [0.43 to 0.53]</td>
<td>Global</td>
<td>High</td>
</tr>
<tr>
<td>Volatile organic compounds</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CH₄</td>
<td>0.16 [0.14 to 0.18]</td>
<td>Global</td>
<td>Low</td>
</tr>
<tr>
<td>Halocarbons</td>
<td>0.34 [0.31 to 0.37]</td>
<td>Global</td>
<td></td>
</tr>
<tr>
<td>Ozone</td>
<td>-0.05 [-0.05 to 0.05]</td>
<td>Continental to global</td>
<td>Med</td>
</tr>
<tr>
<td>Stratospheric</td>
<td>0.35 [0.28 to 0.65]</td>
<td>Continental to global</td>
<td>Med</td>
</tr>
<tr>
<td>Stratospheric water vapour from CH₄</td>
<td>0.07 [0.02 to 0.12]</td>
<td>Global</td>
<td>Low</td>
</tr>
<tr>
<td>Land use</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Surface albedo</td>
<td>0.2 [-0.4 to 0.0]</td>
<td>Local to continental</td>
<td>Med</td>
</tr>
<tr>
<td>Black carbon on snow</td>
<td>0.1 [0.0 to 0.2]</td>
<td>Local to continental</td>
<td>Low</td>
</tr>
<tr>
<td>Black carbon on snow</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Direct effect</td>
<td>-0.5 [-0.9 to -0.1]</td>
<td>Continental to global</td>
<td>Med</td>
</tr>
<tr>
<td>Total Aerosol</td>
<td>0.7 [-1.8 to 0.3]</td>
<td>Continental to global</td>
<td>Low</td>
</tr>
<tr>
<td>Cloud albedo effect</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Linear contrails</td>
<td>0.01 [0.003 to 0.003]</td>
<td>Continental</td>
<td>Low</td>
</tr>
<tr>
<td>Natural</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solar irradiance</td>
<td>0.12 [0.08 to 0.30]</td>
<td>Global</td>
<td>Low</td>
</tr>
<tr>
<td>Total net anthropogenic</td>
<td>1.6 [0.6 to 2.4]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
AR4 adjustment of 20th century simulation

Hindcasts and Forecasts of Global Mean Temperature

Ed Tredger
(PhD thesis, LSE, 2009)

L.A. (“Lenny”) Smith (2009)
personal communication
The uncertainties might be *intrinsic*, rather than mere “tuning problems.”

If so, maybe *stochastic structural stability* could help!

Might fit in nicely with recent taste for “stochastic parameterizations.”

*The DDS dream of structural stability* (from Abraham & Marsden, 1978)
A linear example as a paradigm

Let us first start with a very difficult problem:

Study the “dynamics” of \( \dot{x} = -\alpha x + \sigma t, \quad \alpha, \sigma > 0 \). \( (1) \)

First remarks:

- The system \( \dot{x} = -\alpha x \), i.e. the autonomous part of (1), is dissipative. All the solutions of \( \dot{x} = -\alpha x \) converge to 0 as \( t \to +\infty \).
- Is it the case for (1)? Certainly not! The autonomous part is forced; we even introduce an infinite energy over an infinite time interval: \( \int_{0}^{+\infty} t \, dt = +\infty \! \right) \). Forward attraction seems to be ill adapted to time-dependent forcing.

Goal:

Find a concept of attraction that is:

(i) compatible with the forward concept, when there is no forcing; and
(ii) provides a way to assess the effect of dissipation in some sense.

For that let’s do some computations...
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A tool for classification: stochastic equivalence

- **Stochastic equivalence**: two cocycles $\varphi_1(t, \omega)$ and $\varphi_2(t, \omega)$ are conjugated iff there exists a random homeomorphism $h \in \text{Homeo}(X)$ and an invariant set $\tilde{\Omega}$ of full $\mathbb{P}$-measure (w.r.t. $\theta$) such that $h(\omega)(0) = 0$ and:

$$\varphi_1(t, \omega) = h(\theta(t)\omega)^{-1} \circ \varphi_2(t, \omega) \circ h(\omega);$$ (2)

$h$ is also called **cohomology** of $\varphi_1$ and $\varphi_2$. It is a random change of variables!

- **Motivation**: We would like to measure quantitatively as well as quantitatively the difference between climate models.
\( \phi \) is a random dynamical system (RDS)

\( \Theta(t)(x, \omega) = (\theta(t)\omega, \varphi(t, \omega)x) \) is a flow on the bundle
Stochastic equivalence - Could noise help the classification?

As the noise variance tends to zero and/or the parametrizations are switched off, one recovers the structural instability, as a "granularity" of model space. For nonzero variance, the random attractor \( \{A(\omega)\} \) associated with several GCMs might fall into larger and larger classes as the noise level increases.
We want to perform a classification in terms of stochastic equivalence.

Our first theoretical laboratory is Arnold’s family of diffeomorphisms of the circle:

\[ x_{n+1} = F_{\Omega,\varepsilon}(x_n) := x_n + \Omega - \varepsilon \sin(2\pi x_n) \mod 1 \]
Which paradigm is represented by this family? Why this family?

- Frequency-locking phenomena & Devil’s staircase
- Topological classification of Arnold’s family \( \{ F_{\Omega, \varepsilon} \} \):
  - Countable regions of structural stability,
  - Uncountable structurally unstable systems with non-zero Lebesgue measure!
- Two types of attractors:
  - Periodic orbits in the circle.
  - The whole circle.
Arnold’s tongues and Devil’s staircase
Effect of the noise on topological classification?

\[ \sigma = 0.05 \quad \sigma = 0.10 \quad \sigma = 0.15 \]

Effect of the noise on the PDF of Arnold’s tongue 1/3
Extension of the paradigm - Devil’s quarry

Short description of the deterministic model

- **Dynamics on a 2-D torus:**
  \[
  x_{n+1} = x_n + \Omega_1 - \varepsilon \sin(2\pi y_n), \mod 1
  \]
  \[
  y_{n+1} = y_n + \Omega_2 - \varepsilon \sin(2\pi x_n) \mod 1
  \]

- **Web of resonances & chaos:**
  - Partial resonance \((\Omega_1, \Omega_2)\) are rational and there is one rational relation \(m_1\Omega_1 + m_2\Omega_2 = k \in \mathbb{Z}^*\) with \((m_1, m_2) \in \mathbb{Z}^* \times \mathbb{Z}^*\)
  - Full resonance
  - Chaos with possibly multiple attractors

- **A more realistic paradigm of observed dynamics in the geosciences, and more...**

- **What is the effect of noise in such a context?**

Michael Ghil, Mickaël D. Chekroun, Eric Simonnet, Ilya Zaliapin
A French garden near the castle of La Roche-Guyon
Devil’s quarry for a coupling parameter $\varepsilon = 0.15$: a web of resonances
Devil's Bleachers in a 1-D ENSO Model

Ratio of ENSO frequency to annual cycle

Sample measure supported by the R.A.

Another proj. of the sample measure, “friendlier”

- The next slides are similar, with different noise level $\alpha$ and more I.D....
Sample measure supported by the R.A.

- 1 Billion I.D., and a different color palette!
- Intensity is $\alpha = 0.2$.
- Do you want different noise intensities?
Here $\alpha = 0.4$. The sample measure is approximated for another realization $\omega$ of the noise, starting from 8 billion I.D.

Now more serious stuff is coming...
Sample measures evolve with time.

- Recall that these sample measures are the frozen statistics at a time $t$ for a realization $\omega$.

- How do these frozen statistics evolve with time?

- Action!
Sample measures evolve with time.

- Recall that these sample measures are the frozen statistics at a time $t$ for a realization $\omega$.

- How do these frozen statistics evolve with time?

**Action!**
RDS theory offers a rigorous way to define random versions of stable and unstable manifolds, via the Lyapunov spectrum, the Oseledec multiplicative theorem, and a random version of the Hartman-Grobman theorem.

When the sample measures $\mu_\omega$ of an RDS have absolutely continuous conditional measures on the random unstable manifolds, then $\mu_\omega$ is called a *random SRB measure*.

If the sample measure of an RDS $\varphi$ is SRB, then its a "physical" measure in the sense that:

$$\lim_{s \to -\infty} \frac{1}{t-s} \int_s^t G \circ \varphi(s, \theta_{-s}\omega) x \, ds = \int_{A(\theta_t\omega)} G(x) \mu_{\theta_t\omega}(dx), \quad (3)$$

for almost every $x \in X$ (in the Lebesgue sense), and for every continuous observable $G : X \to \mathbb{R}$.

The measure $\mu_\omega$ is also the image of the Lebesgue measure under the stochastic flow $\varphi$: for each region of $A(\omega)$, it gives the probability to end up on that region, when starting from a volume.
A remarkable theorem of Ledrappier and Young (1988)

1. Ledrappier and Young have proved that, that if the stationary solution, $\rho$, of the Fokker-Planck equation associated to an SDE presenting a Lyapunov exponent $> 0$, has a density w.r.t. the Lebesgue measure, then:

   $\mu_\omega$ is a random SRB measure.

2. This theorem applies to a large class of dissipative stochastic systems, namely the hypoelliptic ones that exhibit a Lyapunov exponent $> 0$: they all support a random SRB measure.

3. Furthermore, we have the important relation:

   \[ \mathbb{E}(\mu_\omega) = \rho, \]

   where $\rho$ is the stationary solution of the Fokker-Planck equation, when the latter is unique.
The Ruelle response formula

Physically, the challenge is to find the trade-off between the physics present in the model and the stochastic parameterizations of the missing physics.

From a mathematical point of view, climate sensitivity can be related to sensitivity of SRB measures.

The thermodynamic formalism à la Ruelle, in the RDS context, helps to understand the response of systems out-of-equilibrium, to changes in the parameterizations (Gundlach, Kifer, Liu).

The Ruelle response formula: Given an SRB measure $\mu$ of an autonomous chaotic system $\dot{x} = f(x)$, an observable $G : X \to \mathbb{R}$, and a smooth time-dependent perturbation $X_t$, the time-dependent variations $\delta_t \mu$ of $\mu$ are given by:

$$
\delta_t \mu(G) = \int_{-\infty}^{t} d\tau \int \mu(dx) X_{\tau}(x) \cdot \nabla_x (G \circ \varphi_{t-\tau}(x)),
$$

where $\varphi_t$ is the flow of the unperturbed system $\dot{x} = f(x)$.
The susceptibility function

- In the case \( X_t(x) = \phi(t)X(x) \), the Ruelle response formula can be written:
  \[
  \delta_t \mu(G) = \int dt' \kappa(t - t')\phi(t'),
  \]
  where \( \kappa \) is called the response function. The Fourier transform \( \hat{\kappa} \) of the response function is called the susceptibility function.

- In this case \( \delta_t \mu(G)(\xi) = \hat{\kappa}(\xi)\hat{\phi}(\xi) \) and since the r.h.s. is a product, there are no frequencies in the linear response that are not present in the signal.

- In general, the situation can be more complicated and the theory gives the following criterion of high sensitivity:

  \( \mathcal{C} \): Poles of the susceptibility function \( \hat{\kappa}(\xi) \) in the upper-half plane \( \Rightarrow \) High sensitivity of the system’s response function \( \kappa(t) \).

- RDS theory offers a path for extending this criterion when random perturbations are considered.
Concluding remarks, II – General

What do we know?
• It’s getting warmer.
• We do contribute to it.
• So we should act as best we know and can!

What do we know less well?
• By how much?
  – Is it getting warmer …
  – Do we contribute to it …
• How does the climate system (atmosphere, ocean, ice, etc.) really work?
• How does natural variability interact with anthropogenic forcing?

What to do?
• Better understand the system and its forcings.
• Explore the models’, and the system’s, robustness and sensitivity
  – stochastic structural and statistical stability!
  – linear response = response function + susceptibility function
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  – linear response = response function + susceptibility function!!
Climatic uncertainties & moral dilemmas

♥ Feed the world today
or…

see also Hillerbrand & Ghil, Physica D, 2008, 237, 2132–2138,
The Biofuel Myth

- Fine illustration of the moral dilemmas (*).
- Conclusion: “festina lentae,” as the Romans (**) used to say..

(**) ~ Han dynasty
