Deterministic and stochastic models of tropical climate variability

by

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5.2 **Initial Estimation**

- **a** Variance explained by each PC.
- **b** Each PC skewness.
- **c** Each PC excess kurtosis. Blue circles denote observed values, red plus signs CAM noise modeled values, and yellow crosses additive noise modeled values.

5.3 **Iterated Estimation**

- **a** Variance explained by each PC.
- **b** Each PC skewness.
- **c** Each PC excess kurtosis. Blue circles denote observed values, Red plus signs CAM noise modeled values, and yellow crosses additive noise modeled values.

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Abstract

The first part of this thesis studies the low-frequency ocean-atmosphere coupling relevant to describe tropical meridional modes using an analytical framework and simple sensitivity experiments. We consider a simple Gill-Matsuno atmospheric model coupled to a thermodynamic slab ocean model to describe the wind-evaporation-sea surface temperature (WES) feedback. We develop an analytical understanding of the coupled processes responsible for the growth and propagation of meridional mode-like structures by projecting the ocean and steady-state atmosphere equations onto parabolic cylinder functions. By doing this the coupling simplifies to the non-normal interaction of different sea surface temperature (SST) modes mediated by atmospheric Kelvin and Rossby waves. Under a homogeneous coupling the system simplifies to independent, coupled equatorially symmetric and antisymmetric (meridional mode-like) modes, with the following major findings: the non-normal interaction is responsible for a propagation from high latitudes to low latitudes, and this process works similarly for equatorially symmetric and antisymmetric coupled structures. That similarity breaks near the equator where the antisymmetric structure grows through a positive feedback mediated by the first antisymmetric Rossby wave, while the symmetric structure is subjected to a negative WES feedback arising from the atmospheric Kelvin wave interaction. Taken all together this explains why the WES feedback preferentially sustains meridional mode-like structures.
The second part of this thesis deals with the stochastic parameterization of tropical climate variability. Although usually seen as arising from slow non-linear interactions, deviations from Gaussianity in the Tropical Pacific may originate through fast variability that may be parameterized as multiplicative noise. We develop and apply a stochastic parameterization that accounts for the Tropical Pacific SSTs deviations from Gaussianity. This model, termed CAM-LIM retains the successful elements of the regular Linear Inverse Model (LIM) but it improves in the representation of the higher order statistical moments. The CAM-LIM is used to correctly model the marginal and conditional probability density functions of the different SST principal components. It is found that the model is successful at describing the observed deviations from Gaussianity such as: it rightfully describes the ratio between positive and negative Niño events, and the frequency of extreme Niño events as well.
Chapter 1: Introduction

The most well known ocean-atmosphere coupled mode is the El Niño Southern Oscillation (ENSO) (Bjerknes 1969). This mode affects the climate all over the world at interannual timescales, and its study is a very active research area. In addition to ENSO, for which the ocean-atmosphere coupling involves active ocean dynamics, there are other coupled modes of variability where the ocean takes a more passive role known as meridional modes. These kind of modes, such as the Pacific Meridional Mode (PMM) (Chiang and Vimont 2004), and Atlantic Meridional Mode (AMM) (Servain et al. 1999; Chiang and Vimont 2004) are forced by trade wind variations in their extratropical/subtropical flanks, propagate equatorwards and maintain their variance through ocean-atmosphere coupled flux variations. In this way they are able to bridge the extratropics with the tropics.

Tropical climate variability, by definition, varies at low frequencies. Nonetheless, the low frequency phenomena may be affected by variability at all timescales, including variability with a decorrelation timescale short compared to the relevant low frequency timescale. In that particular case the input of rapidly decorrelating variability, may in principle be parameterized stochastically. This dissertation is separated in two parts. First (chapters 2 and 3), it will take a theoretical and analytical look at the variety of tropical modes that arise through internal ocean-atmosphere deterministic thermal coupling. While the second part (chapters 4 and 5) deals with the stochastic parametrization of tropical climate vari-
ability using inverse methods. Although both of these models are relevant in explaining our
tropical climate, their scope and methods differ greatly. Below we will present the relevant
background needed to contextualize the research presented in chapters 2 to 5.

1.1 Tropical Variability coupled through the WES feedback

The Wind-Evaporation-SST (WES) feedback is a very relevant part of our climate system. This feedback explains coupled modes such as the Atlantic Meridional Mode (AMM) (figure 1.1a) and Pacific Meridional Mode (PMM, figure 1.1b) (Chiang and Vimont 2004). As shown in figure 1.1 these two coupled modes share several similarities. Both modes consist of a meridional dipole in SST with an associated mass transport from negative to positive SST anomalies. In addition, the precipitation anomalies associated to them are comparable and consist of an antisymmetric meridional dipole (figures 1.1c and 1.1d). However, differences can also be found. The AMM is an equatorially antisymmetric structure for most of the year (Smirnov and Vimont 2012) with an important inter-hemispheric low level mass flow. On the other hand, there is an important degree of equatorial symmetry in the PMM, and consequently the inter-hemispheric low level mass flow is reduced.

The WES feedback (Xie and Philander 1994) was introduced to explain observed equatorially antisymmetric structures. In that case to explain the antisymmetry of the mean state in the eastern/central Pacific. This feedback has usually been explained in the following way: An initial SST inter-hemispheric gradient generates an atmospheric circulation that includes winds blowing from the cold hemisphere into the warm hemisphere. This anomalous circulation relaxes the trade winds in the warm hemisphere and reinforces them in the cold hemisphere. In this way the wind-induced ocean latent heat release is modified reinforcing the original inter-hemispheric SST gradient. In this thesis we demonstrate that although
the WES feedback coupled variability is enhanced for antisymmetric structures, there are regimes in the parameter space where equatorially symmetric WES feedback generated structures may be important. The physical processes that sustain these equatorially symmetric modes are explored in depth in chapter 3.

In chapters 2 and 3 we study the internally coupled modes that arise through a linearized WES feedback coupling. In mathematical terms a system like this may be represented as

\[
\frac{d\vec{x}}{dt} = \mathbf{M}\vec{x} + \text{noise} + \text{external forcing}. \tag{1.1}
\]

Here the matrix \( \mathbf{M} \) is a stable linear operator that contains the WES feedback interaction (see equations 2.1 and 3.20), and \( \vec{x} \) is the state vector of the system. Since we will consider \( \mathbf{M} \) to be stable, the noise input/external forcing on the system is necessary to maintain variance. In chapters 2 and 3 we will focus on the internal modes that arise from the deterministic coupling interaction. That is, we will focus on the first term in the right hand side of equation 1.1. Variations of equation 1.1, not only restricted to WES feedback interactions, will be broadly studied in chapters 4 and 5.

Although \( \mathbf{M} \) is a stable operator – i.e. all its eigenvalues are negative, no runaway growth of the system is possible –, the deterministic dynamics encapsulated by \( \mathbf{M} \) may temporally grow the system variance if \( \mathbf{M} \) is non-normal (i.e. \( \mathbf{M} \neq \mathbf{M}^\dagger \), see appendix F). There are optimal structures that will maximize this transient growth of variance through their deterministic evolution. These growing structures will appear as dominant patterns of covariability. Chapter 2 analyzes this transient process under structural variations in the mean state, and consequently, the WES feedback coupling. Chapter 3 goes deeper into the causes of the transient growth and propagation mechanisms of structures coupled through this feedback.

From here we take a deterministic view on the WES feedback coupling, this is the coupled
response is slow enough that it can be resolved and treated with standard methods. External forcing on meridional mode variability may be modulated in nature by low frequency extratropical modes, but an important part of that forcing occurs at more rapid timescales compared to the ones of the AMM/PMM. Thus, they are subject to be modeled stochastically. In addition to this the low-frequency SST state may affect the rapid wind variability response giving rise to “stochastic coupling” (Sura et al. 2005). We make no attempt herein to explicitly model this contributions in the WES feedback context. These considerations connect naturally to the second part of this thesis.

1.2 Stochastic Models of Tropical Climate Variability

The climate system is comprised of phenomena acting at different timescales. If we could resolve all our forcings and we had infinite available computational power, in principle we could run a deterministic model that accounts for all the timescales interactions. In practice this may not ever be possible. There arises the need to “parameterize” unresolved phenomena in terms of what we know explicitly. One ways these unresolved forcings may be treated is through a “stochastic parameterization”.

To motivate this further consider the evolution equation of the following “slow” low-frequency vector \( \bar{x} \), with the timescales of the different interactions explicitly separated as (Sura et al. 2005, equation 1):

\[
\frac{d\bar{x}}{dt} = \mathbf{L}\bar{x} + \mathbf{N}_1(\bar{x}, \bar{x}) + \mathbf{N}_2(\bar{x}, \bar{y}) + \mathbf{N}_3(\bar{y}, \bar{y}) + \bar{F},
\]

(1.2)

where \( \bar{y} \) represents the non-temporally resolved (fast) variability. In this equation \( \mathbf{L}\bar{x} \) represents the linear dynamics, \( \mathbf{N}_1 \) the non-linear slow-slow interactions, \( \mathbf{N}_2 \) slow-fast interactions, and \( \mathbf{N}_3 \) fast-fast interactions. The vector \( \bar{F} \) represents low-frequency external forcing.
To make things simpler let’s consider a one dimensional version of (1.2), (with no external forcing, it can always be added back at the end) and recast it as (Penland 1996 equation 8)

\[
\frac{dx}{dt} = F(x, t) + S(x, t)\hat{\eta}(t). \tag{1.3}
\]

Here \(F\) and \(S\) are deterministic quantities, and \(\hat{\eta}(t)\) is a fast variable with a short decorrelation time compared to the characteristic timescales of \(F\) and \(S\). Here we have made the equivalence \(F(x, t) = Lx + N_1\) and \(S\hat{\eta}(t) = N_2 + N_3\). In the limit where the \(\hat{\eta}\) autocorrelation function goes to a \(\delta\) function, equation 1.3 approximates weakly to the Stratonovich stochastic differential equation (Papanicolaou and Kohler 1974; Penland 1996; Ewald and Penland 2009)

\[
\frac{dx}{dt} = F(x) + S(x)\eta, \tag{1.4}
\]

Approximating the autocorrelation function of the fast variable to a \(\delta\) function combines the effect of the weakly correlated fast fluctuations such that the Central Limit Theorem (Papanicolaou and Kohler 1974) applies, and as a consequence, \(\eta\) is a Gaussian white noise process with the following properties (Penland 2003):

\[
< \eta(t) > = 0
\]

\[
< \eta(t)\eta(0) > = \delta(t), \tag{1.5}
\]

where the symbol < > represents the expectation value. The integration in time of Gaussian white noise (where we have defined \(dW = \eta dt\))

\[
W(t) = \int_0^t dW(t) \tag{1.6}
\]

is known as a Wiener process, Brownian motion or continuous random walk, and has a red
In chapters 4 and 5 we will make use of the multivariate analogous of (1.4)

\[
\frac{d\vec{x}}{dt} = \mathbf{F}(\vec{x}) + \mathbf{S}(\vec{x})\vec{\eta}.
\]  

(1.7)

Here \(\vec{\eta}\) represents a vector of independent Gaussian white noise processes. The interested reader may consult Ewald and Penland 2007 and the references within for a rigorous derivation of the stochastic limit in a system with a clear separation of timescales.

1.2.1 Stochastic approximation applied to the Tropical Pacific

In chapter 5 we use (1.7) applied to the Tropical Pacific. First, a few approximations are in order. Penland and Sardeshmukh 1995 shows that the Tropical Pacific SST deterministic variability can be well encapsulated by a linear operator. In addition there have been some efforts to incorporate slow-nonlinearities to this kind of models with marginal improvements in terms of predictability (Kondrashov et al. 2005; Chen et al. 2016). Consequently in our case we will approximate \(\mathbf{F}(\vec{x})\) as \(\mathbf{F}_{\text{lin}} = (\mathbf{L} + \mathbf{N}_{\text{lin}})\vec{x}\) where \(\mathbf{N}_{\text{lin}}\vec{x}\) accounts for the linearization of \(\mathbf{N}_1\) in (1.2). We will study two stochastic parameterizations, the first one just accounts for fast-fast variability (\(\mathbf{N}_2 = 0\), what in this case corresponds to an Ornstein-Uhlenbeck process), and the second one in addition considers a linear dependence of the stochastic input on the slowly varying SST state (\(\mathbf{N}_2 \neq 0\)).

In the simplest case the SSTs (denoted as \(\vec{\eta}\)) evolve following:

\[
\frac{d\vec{x}}{dt} = \mathbf{M}\vec{x} + \mathbf{B}\vec{\eta}.
\]  

(1.8)

The matrix \(\mathbf{B}\) does not have arbitrary values, it depends on the sensitive balance between
deterministic and stochastic terms. This implies that although the noise is uncorrelated in time, it does have a spatially coherent structure. We rewrite the previous equation as

$$\frac{d\tilde{x}}{dt} = M\tilde{x} + \tilde{R}. \quad (1.9)$$

where $\tilde{R}$ is the remainder after we account for the linearized deterministic dynamics, and it is the term that is stochastically parameterized in (1.8) by $B\eta$. This determination of $\tilde{R}$, which can considered as one particular realization of the noise, is calculated as

$$\tilde{R} = (\frac{d}{dt} - M)\bar{x} \quad (1.10)$$

and has been diagnosed in the Tropical Pacific using weekly SST and thermocline data from 1982 to 2015 (Thomas, Martinez-Villalobos, Penland, Newman and Vimont, in prep.). An example of this calculation is shown in figure 1.2.

In the study presented here we take a different and complementary approach. We determine the stochastic parameterization by requiring the stochastic output to be balanced by the deterministic dynamics such that the statistics of the system are stationary. That is the mean, covariance, skewness and kurtosis structure of the system is fixed, and we determine the stochastic input on the system to enforce that.
Figure 1.1: a PMM pattern. b AMM pattern. c Precipitation anomalies associated to the PMM. d Precipitation anomalies associated to the AMM. From Chiang and Vimont 2004.
Figure 1.2: Noise forcing time series associated with 6-month Central Pacific (top) and Eastern Pacific (bottom) optimal initial conditions. Calculated as the projection of $R(t)$ onto the spatial structures of the CP/EP optimal (Vimont et al. 2014) initial conditions (Thomas et al. 2016).
Chapter 2:

**The Role of the Mean State in Meridional Mode Structure and Growth**

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**Abstract**

This study uses a simple linear coupled model to investigate the role of the WES feedback and ITCZ mean states in meridional mode variability. Optimal structures that maximize transient growth are calculated for mean states characteristic of boreal spring and boreal fall in the tropical Atlantic. During boreal spring the leading optimal structure is a zonal mode that propagates westward and does not resemble observed meridional mode. In contrast, the leading optimal structure during fall is a sea surface temperature (SST) monopole over the northern hemisphere (NH) that propagates equatorward and westward and that closely matches meridional mode variability during this season. It is found that the boreal fall optimal growth greatly exceeds growth of the corresponding optimal during boreal spring, despite the observed boreal spring peak in Atlantic meridional mode variance.
Sensitivity studies are used to explore the role of northern or southern hemisphere initial conditions, ITCZ width, and ITCZ location in meridional mode growth and structure. It is found that: (i) growth is favored for optimal structures that originate in the northern hemisphere, especially for boreal fall mean states; (ii) for symmetric mean states, equatorially symmetric structures maximize growth under narrow ITCZ configurations, and anti-symmetric structures maximize growth under wider ITCZ configurations; (iii) for anti-symmetric mean states (and realistic ITCZ width) growth is maximized when the ITCZ is located off of the equator. The implications of these findings are discussed.
2.1 Introduction

Meridional modes of climate variability are the dominant source of ocean / atmosphere variability in the Tropical Atlantic [the Atlantic Meridional Mode (AMM); Moura and Shukla (1981); Chang et al. (1997); Servain et al. (1999); Chiang and Vimont (2004)], and play a secondary role to the El Niño / Southern Oscillation (ENSO) as a source of ocean / atmosphere variability in the Pacific [the Pacific Meridional Mode (PMM); Chiang and Vimont (2004)]. Meridional mode variations have also been studied in the South Pacific (Zhang et al. 2014a), the western Pacific (Wang et al. 2012), and the Indian Ocean (Wu et al. 2008). Meridional modes are characterized by anomalous meridional SST gradient that drives an anomalous interhemispheric boundary layer flow from cold to warm waters. Associated with this anomalous boundary layer flow is an ITCZ shift in the direction of the anomalously warm hemisphere (Hastenrath and Heller 1977). The ubiquity of meridional mode variations in the tropical climate system motivates further understanding of the role of the mean state in meridional mode structure and variation.

Meridional modes of climate variability emerge due to a positive feedback between wind, evaporation, and SST [the WES feedback; Xie and Philander (1994); Chang et al. (1997)]. The WES feedback has been explained in the following way: a SST anomaly in one hemisphere creates a cross-equatorial, interhemispheric SST and pressure gradient that drives a meridional flow from the cold hemisphere to the warm one. The Coriolis force deflects the meridional flow to the right in the Northern Hemisphere, and to the left in the Southern Hemisphere, which would reenforce the mean easterly trades in the cold hemisphere and reduce the mean easterly trades in the warm hemisphere. This change in wind speed modulates the evaporative latent heat flux at the surface, reinforcing the original SST gradient. Vimont (2010) show that under the quasi-geostrophic approximation neither the cross
equatorial SST gradient nor the reversal in sign of the Coriolis parameter at the equator is necessary for growth via a WES feedback. Instead, a meridional variation in mean absolute vorticity is sufficient to produce instability and westward propagation of meridional mode structures. Additional feedbacks involving vertical mixing in the ocean (Xie and Philander 1994) and cloud radiative effects (Tanimoto and Xie 2002; Xie and Saito 2001; Evan et al. 2013) have been shown to influence meridional mode variation.

The mean state can influence meridional mode variations through the ITCZ structure, the mean trade wind strength and location, and through variation in stochastic forcing that is likely required for meridional mode variations in nature (Xie 1999; Vimont 2010). The mean ITCZ structure can affect how the atmosphere responds to tropical SST variations via either deep heating (Gill 1980; Zebiak 1986; Battisti et al. 1999) or boundary layer convergence and ventilation (Lindzen and Nigam 1987; Battisti et al. 1999). Mean trade winds influence the strength of the WES feedback via the role of wind speed variations on evaporation (Xie and Philander 1994; Czaja et al. 2002; Vimont et al. 2009; Vimont 2010; Zhang et al. 2014b). The WES feedback has also been implicated in explaining the mean ITCZ state asymmetry in the first place (Xie and Philander 1994; Xie and Saito 2001; Takahashi and Battisti 2007). The study of the ITCZ mean state and variability remains an area of active research (Tomas and Webster 1997; Toma and Webster 2009a,b; Frierson et al. 2013; Schneider et al. 2014; Bischoff and Schneider 2014). External / stochastic forcing influences meridional mode variance primarily through geographic variations in trade wind variations and associated surface heat flux to the respective subtropical oceans, especially during winter (Nobre and Shukla 1996). ENSO and the North Atlantic Oscillation (NAO) have both been shown to force the AMM (Xie and Tanimoto 1998; Czaja et al. 2002; Chiang and Vimont 2004; Kossin and Vimont 2007; Penland and Hartten 2014), while in the Pacific the North Pacific Oscillation (NPO) (Rogers 1981; Linkin and Nigam 2008) has been shown
as a source of external forcing for the PMM (Chiang and Vimont 2004; Chang et al. 2007; Vimont et al. 2009).

Meridional Mode variability over the Atlantic possesses a strong seasonality with peak variance in boreal Spring. During this season, meridional mode variability corresponds to the so called “Atlantic SST dipole”, and consists of an inter-hemispheric anti-symmetric SST pattern, although the anti-coherence has been questioned (Houghton and Tourre 1992; Enfield et al. 1999). The boreal spring maximum in meridional mode variability appears to result from enhanced atmospheric forcing during the preceding season (Czaja et al. 2002; Czaja 2004). However, the contribution of the seasonal cycle (ITCZ location and strength) to the seasonality of meridional mode dynamics is not clear. Meridional mode variability has been less extensively studied during boreal summer and fall when, in the Atlantic, the AMM structure consists mostly of single SST “monopole” (in contrast to the more familiar equatorially anti-symmetric dipole) over the Northern Hemisphere subtropics, with little activity over the Southern Hemisphere (Smirnov and Vimont 2011). The relative roles of internal dynamics and external forcing are also not clear during this season, although compared to Spring, external atmospheric forcing is not at its peak.

This study investigates the role of mean state asymmetry on meridional mode structure and growth, including both the effect of asymmetry on internal dynamics and on external forcing. This paper extends the scope of Vimont (2010) who investigate transient growth of meridional mode disturbances under an equatorially symmetric mean state. The remainder of this paper is organized as follows. Section 2 will describe the model and methods utilized. Section 3 will present the optimal results under symmetric and asymmetric ITCZ states, and some sensitivity tests. Finally section 4 will present the conclusions.
2.2 Model and Analysis

2.2.1 Model

We use a linear dynamical model to investigate growth of disturbances that resemble meridional modes in the tropics. The model we use is the same as used by Vimont (2010), and consists of a Gill-Matsuno model of the atmosphere coupled with a thermodynamic “slab” ocean model:

\[
\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \\ \phi \\ T \end{pmatrix} = \begin{pmatrix} -\epsilon & \beta y & -\frac{\partial}{\partial x} & 0 \\ -\beta y & -\epsilon & -\frac{\partial}{\partial y} & 0 \\ -c^2 \frac{\partial}{\partial x} & -c^2 \frac{\partial}{\partial y} & -\epsilon & -K_q(y) \\ \alpha(y) & 0 & 0 & -\epsilon + \gamma \nabla^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ \phi \\ T \end{pmatrix}. \tag{2.1}
\]

The model and constants are described fully in Vimont (2010) and Table 2.1. Note that the ocean is coupled to the atmosphere via the WES parameter \( \alpha(y) \) \([\alpha(y) = \hat{\alpha}(y)/(\rho_o c_o H_o)]\) and the atmosphere is coupled to the ocean via deep heating which is linearly proportional to the SST anomaly via the parameter \( K_q(y) \).

The non-dimensionalized (Vimont 2010) version of (2.1) forms our coupled model, which is spectrally decomposed into a single zonal harmonic \((2\pi/k = 120^\circ)\) and discretized in the meridional direction using centered differencing. A sponge layer of enhanced linear damping is applied to the meridional boundaries to eliminate spurious boundary waves. Additional details of the model can be found in Vimont (2010). Note that this model does not include other physical processes that may influence tropical atmosphere/ITCZ/SST interactions such as inertial instability (Tomas and Webster 1997; Toma and Webster 2009a,b) or realistic Hadley cell variations (Frierson et al. 2013).
2.2.2 Background States

The background mean state affects the model equations (1) through the location and intensity of the inter tropical convergence zone (ITCZ) which affects the meridional structure of $K_q(y)$, and the surface zonal winds which affect the structure and amplitude of the WES parameter $\alpha(y)$. The structure of these two parameters determines coupled system growth, which is a function of the effective coupling $K_q(y)\alpha(y)$, everything else held constant (Hirst 1986; Appendix A herein). Here, we find that variations in $K_q$ have a larger impact than variations in the WES parameter, and hence may play an important role in the structure and evolution of meridional modes.

2.2.2.1 $K_q$ and ITCZ mean states

In our model the atmosphere is coupled to the ocean via a heating that is proportional to the local SST anomaly. Following Battisti et al. (1999) we parameterize heating to be proportional to the SST anomaly only in regions of mean low-level moisture convergence. A good proxy for mean moisture convergence in the boundary layer is given by $P - E$

$$- \int_0^h \nabla \cdot (\rho_w \mathbf{u}) dz \approx P - E,$$

(2.2)

where $h$ is the boundary layer top height. A zonal average over the tropical Atlantic of $P - E$ is used to determine regions of mean moisture convergence and provides a meridional structure for $K_q(y)$. Figure 2.1 shows the location and strength of the ITCZ (parameterized by $K_q$) at two different times of the year for the tropical Atlantic. Fig. 2.1b represents the most symmetric ITCZ state ($K_q$ in Fig. 2.1c), which is constrained in shape by the average of $P - E$ in February-March-April. Fig. 2.1e represents the most anti-symmetric ITCZ mean state ($K_q$ in Fig. 2.1f), which is constrained in shape by the average of $P - E$ on
August-September-October.

2.2.2.2 The WES parameter $\alpha(y)$ and surface winds

The WES parameter $\alpha(y)$ is diagnosed by calculating the linearized change in latent heat flux per unit of zonal wind change (Czaja et al. 2002; Vimont 2010). Under the bulk approximation for surface latent heat flux, the WES parameter can be written in terms of the mean latent heat flux $LH$, the mean zonal wind $\bar{u}$, and the mean wind speed $\bar{w}$:

$$-\alpha(x, y) \equiv LH \frac{\bar{u}}{\bar{w}^2}. \quad (2.3)$$

The meridional structures for the WES feedback parameter are shown in Fig. 2.2 for the 3-month averages that correspond to the ITCZ structure in Fig. 2.1. The solid curve shows the zonal average of the WES feedback for February-March-April, and the dashed curve the analogous term for August-September-October conditions over the tropical Atlantic. Note that the seasonal variation of the WES parameter is much smaller than that of the mean heating (Fig. 2.1).

2.2.3 Transient Growth Analysis

In the present manuscript, we focus on transient growth of the coupled model rather than growth of individual modes. While Farrell and Ioannou (1996) provides a more complete distinction between the two processes, we provide a brief explanation here. In short, modal growth (growth of a specific eigenvector of the system) is most appropriate for normal systems (which have orthogonal modes), or as $t \to \infty$. However, in general (and in the case considered herein) the model dynamics are non-normal, which means that the eigenvectors are not orthogonal. In this case, even if all eigenvectors are damped, growth of particular
structures can still occur over a finite time period due to constructive interference between the eigenvectors. In that case, continual excitation of the system by stochastic forcing is required to maintain variance. This paradigm appears to be more appropriate for meridional mode behavior in models, and in nature (see discussion in Introduction). We summarize the distinction here.

The coupled model (2.1) can be written as $\partial \Phi / \partial t = M\Phi$ where $\Phi$ is the state vector $(u, v, \phi, T)$ and the matrix $M$ is the dynamical system matrix. We can spectrally decompose $M$ by assuming a modal solution of the form $\Phi_j(x, y, t) = \Phi_j(y) \exp \left[ i(kx - \omega_j t) \right]$ in which case we obtain the eigenvalue problem:

$$M\Phi_j = -i\omega_j \Phi_j. \quad (2.4)$$

The resulting eigenfunctions of $M$ are the “modes” of the system. For normal systems (in which case all eigenvectors are orthogonal), or as $t \to \infty$, growth is dominated by the least stable eigenvector of the system (Farrell and Ioannou 1996). However, in our case $M$ is non-normal, and hence the modes are not orthogonal. This implies that growth can occur over a finite time period even if all eigenvectors are damped, due to the constructive interference of the eigenvectors.

The non-normality of $M$ implies that transient growth may occur over a finite time period even if all of the individual modes are damped. Transient growth is diagnosed by solving (2.1):

$$\Phi(t - t_0) = \exp \left[ M(t - t_0) \right] \Phi(t_0) \equiv G_\tau \Phi(t_0) \quad (2.5)$$

where $G_\tau$ is the Green’s function, which is a function of lag $\tau = t - t_0$. Growth over time $\tau$
is measured by:

$$\lambda_r = \frac{|\Phi(t - t_0)|^2_N}{|\Phi(t_0)|^2_L} = \frac{\Phi^\dagger(\tau)N\Phi(\tau)}{\Phi^\dagger(0)L\Phi(0)} = \frac{\Phi^\dagger(0)G_r^\dagger N\Phi(0)}{\Phi^\dagger(0)L\Phi(0)}$$  \hspace{1cm} (2.6)$$

where subscripts $N$ and $L$ indicate that the norm is taken under a specified final and initial norm kernel respectively and $N$ and $L$ are the specified final and initial norm kernels in the discretized model. Without loss of generality we can maximize growth of the numerator in (2.6) under the constraint that the initial condition have length 1 under its respective initial norm. The Lagrangian function $L$ condenses that information:

$$L(\Phi, \lambda) = \Phi^\dagger(0)G_r^\dagger N\Phi(0) - \lambda_r[\Phi^\dagger(0)L\Phi(0) - 1]$$  \hspace{1cm} (2.7)$$

By taking the derivative of (2.7) with respect to elements of the state vector $\Phi(0)$ and setting the result equal to zero we obtain the generalized eigenvalue equation:

$$G_r^\dagger N\Phi(0) = \lambda_r L\Phi(0)$$  \hspace{1cm} (2.8)$$

Here $p_r$ are the initial structures that optimize growth over a time $\tau$ and $\lambda_r$ is the growth under the specified final and initial norms. The evolved optimal structures at time $\tau$ can be found by simply calculating

$$q(\tau) = G_r p_r.$$  \hspace{1cm} (2.9)$$

We will refer to the structures represented by $p_r$ as optimal initial structures, or simply “optimals”, and structures in $q(\tau)$ as the optimal final structures.

Equation 2.6 indicates that growth is norm-dependent. The non-dimensionalization of (2.1) however, results in an arbitrary norm because in our case, the choice of a reference SST for non-dimensionalization is arbitrary. In practice, we are interested in coupled struc-
tures with time scales that are long compared to the atmospheric damping time scale (2 days). For this reason we can consider growth of SST only, because for time scales of interest the atmosphere will effectively be in equilibrium with the SST. To evaluate our choice of non-dimensional parameters, and to provide an illustration of the use of a final norm, we re-evaluate transient growth under the equatorially symmetric case considered by Vimont (2010). We compare results calculated under a specified final “SST norm” \( N_{\text{SST}} = \text{diag}(\epsilon, \epsilon, \epsilon, 1) \) where \( \epsilon << 1 \), with results in Vimont (2010) that were calculated under a Euclidean final norm (in the non-dimensionalized system) in which \( N = 1 \), the identity matrix. Here, we use \( \epsilon = 10^{-9} \) rather than \( \epsilon = 0 \) for numerical stability. Our construction of the SST norm emphasizes growth of SST variance.

Figure 2.3 left column shows the initial and final 140 day optimals using \( N_{\text{SST}} \). Comparing to Fig. 7 in Vimont (2010) (135 day optimal) we can see that the structures are essentially the same. Also the growth rates (Figure 6 in Vimont, 2010) of the first three optimals are, at a good precision, unchanged (not shown). Figure 2.3e shows the Euclidean growth for the 140 day Euclidean and SST optimals. Both growth curves are nearly identical. This is because our non-dimensionalization gives much more relative weight to growth of SST anomalies than of atmospheric fields. In the remainder of the paper we will simply use \( N = 1 \), the identity matrix.

To illustrate the effect of varying initial norms, we calculate optimal initial and final conditions under a “Northern Hemisphere” norm in which initial conditions are constrained to be north of the equator. This is achieved by setting \( L_{NH} = \text{diag}(\epsilon H(y) + H(-y)) \) where \( H(y) \) is the Heaviside function, and again \( \epsilon << 1 \) and is not equal to 0 for numerical stability. This norm kernel penalizes an initial condition in the Southern Hemisphere, and effectively allows unconstrained initial conditions in the Northern Hemisphere\(^1\). Optimal initial and

\(^1\)For the same argument, an SST initial norm does not make physical sense for long lead times, as it would penalize initial SST anomalies and allow unconstrained atmospheric initial anomalies. For long lead
final conditions calculated under the Northern Hemisphere initial norm and Euclidean final norm are shown in Fig. 2.3 right panel. Results show an initial condition with amplitude in the Northern Hemisphere only (see appendix B). The response in this case is qualitatively different than that of the Euclidean initial norm in Fig. 2.3. In particular, the response is asymmetric in that it includes both equatorially symmetric and anti-symmetric (e.g. a dipole) structures in SST. This is expected given the asymmetric initial condition.

2.3 Structure and growth under asymmetric mean states

In this section we calculate growth and evolution of optimal structures under various mean state geometries. In Section 2.3.1, we analyze growth under “observed” mean states for boreal Spring and Fall and find that a symmetric (asymmetric) structure produces optimal growth under the Spring (Fall) conditions. We investigate the role of mean-state asymmetry and ITCZ width in Section 2.3.2, and discuss differences in the structures and growth in Section 2.3.3.

2.3.1 Optimal Structures in Boreal Spring and Fall

In this section we will calculate the optimal structures that maximize transient growth over boreal spring and fall, which are the seasons characterized by extreme mean locations of the ITCZ in the Atlantic. To address the asymmetry between the northern hemisphere (NH) and southern hemisphere (SH) extratropical forcing on meridional mode variability, we also calculate the optimals and growth rates that arise when we constrain the optimal initial conditions to either the northern or southern hemisphere. The results shown in this section should hold qualitatively well over the eastern Pacific as the mean WES and ITCZ states...
there are similar to the Atlantic.

Fig. 2.4 shows the 140 days optimal initial and final structures under boreal spring mean states. Fig. 2.4a depicts the unconstrained optimal initial condition, and Fig. 2.4c shows the case in which the initial condition is constrained to the NH. The unconstrained optimal (Fig. 2.4a) is nearly symmetric about the equator (it would be perfectly symmetric if both $\alpha$ and $K_q$ were exactly symmetric), both at the beginning and during its evolution, and propagates westward. Growth of this structure maximizes at 140$d$. The optimal final condition strongly resembles the least stable eigenvector of the system (not shown). We note that the least stable eigenvector of the dynamical system matrix is damped, so any growth must occur through non-normality. The other eigenvectors of the dynamical matrix $M$ are heavily damped for this mean state, so by 140$d$ the remaining modes (which contribute to the non-normal growth prior to 140$d$) have effectively decayed and the least stable eigenvector is left to decay at its own rate.

When the initial condition is constrained to the northern hemisphere (Fig. 2.4c) the optimal is just the northern half of the unconstrained initial condition (Fig. 2.4a, see appendix II). Like the unconstrained optimal, the NH constrained optimal grows into a structure (Fig. 2.4b, and d) that strongly resembles the least stable eigenvector of the system in 140$d$. The SH constrained initial condition (not shown) is very similar in structure to the mirror image of the NH constrained initial condition, although the NH one grows more than its SH counterpart (fig 2.4e). The WES feedback is stronger in the northern hemisphere (fig. 2.2) and the ITCZ is centered slightly north of the equator (fig. 2.1c) on average over FMA, which results in a stronger effective coupling over the northern hemisphere tropics, and a larger growth rate for the NH constrained initial condition. The SH constrained optimal decays at a slower rate than the 120$d^{-1}$ linear oceanic damping rate, implying a positive role of the WES feedback in enhancing SST variance.
The leading optimal does not resemble the anti-symmetric AMM structure that dominates variability in the tropical Atlantic during boreal spring. However, we include a discussion of this structure as similar modes do emerge in other coupled models (Noguchi 1998; Xie et al. 1999). We speculate that this optimal could be related to a thermally coupled Walker mode that is observed in more complex models when the atmosphere is coupled to a thermodynamic “slab ocean” via moisture and heat fluxes (Clement et al. 2011). This optimal also looks similar to one of the unstable modes found in Hirst (1986) (figure 9b there) when the SST equation includes mean zonal SST advection by an anomalous zonal current, but no thermocline feedbacks. In that case, the mean zonal advection produces an effect similar to our WES feedback, leading to similar structure and propagation characteristics.

Over boreal fall the mean state (Figs. 2.1 and 2.2) and resulting optimal structures (Fig. 2.5) are quite different. The unconstrained leading optimal is asymmetric (i.e. a combination of a symmetric and anti-symmetric function) about the equator, with most of the amplitude in the NH. The initial condition starts almost exclusively in the NH, and evolves westward and equatorward over time, becoming more symmetric in the process (Fig. 2.5a and b). This type of “non-dipole” behavior is in accord with the Atlantic Meridional Mode variability observed during boreal summer and fall (see Fig. 2 of Smirnov and Vimont 2011 ). When the initial condition is constrained to the NH, the optimal initial and final conditions (not shown) are virtually identical to the unconstrained optimal, and so is the growth rate (Fig. 2.5c). Note that for both the unconstrained and constrained optimal, growth rates for the ASO mean state conditions are nearly double that of the FMA unconstrained optimal, despite similar WES parameters, and similar width and amplitude of the ITCZ (and hence $K_q$). The explanation for this behavior is traced to atmospheric wave dynamics and will be addressed in section 2.3.3.

During boreal fall the mean ITCZ is completely contained in the NH tropics, implying
that convective support for coupled ocean-atmosphere interaction is asymmetric. Any initial condition in the southern hemisphere would quickly die off because the system is uncoupled there (in this model) during this season. We note that the SH behavior is a limitation to our study, as these variations may be coupled through boundary layer adjustment (Lindzen and Nigam 1987; Battisti et al. 1999; Chiang et al. 2001) in nature. Optimals constrained to start in the SH are not well separated and show very little large scale structure (not shown). Any SH constrained initial optimal decays at a rate close to the linear oceanic damping rate as can be seen in Fig.2.5c for the leading SH optimal.

### 2.3.2 The role of ITCZ location and width

In this section we explore how the mean state affects the growth and symmetry of the optimal response under a symmetric $K_q$ mean state. This is motivated by the different response found in Vimont (2010) compared to our result over boreal spring (Fig. 2.4), and the mismatch between the leading symmetric optimal and the dominant structure of observed ocean-atmosphere variability in the Atlantic. In both cases the mean heating and WES feedback states are (almost) symmetric. Vimont (2010) finds an equatorially anti-symmetric optimal response, while our boreal spring response is symmetric. Are there regimes in which the response will be confined to the equator, with no associated inter-hemispheric boundary layer flow? Will a traditional dipolar anti-symmetric optimal response be possible under other circumstances?

To explore these questions we investigate the change in structure and growth of the leading symmetric and anti-symmetric optimal responses as a function of the width and location of the coupling regions, in particular the $K_q$ parameter. We look at growth rates
and initial and final structures for profiles of $K_q(y)$ that are parameterized as:

$$K_q(y) = \exp \left[ - \left( \frac{y - y_0}{y_w} \right)^4 \right]$$

(2.10)

where $y_0$ is the central ITCZ location and $y_w$ is a characteristic width of the ITCZ. We use this shape since it most closely resembles the profiles constrained by moisture convergence in boreal spring and fall (see Fig. 2.1). We begin by examining growth for a symmetric ITCZ ($y_0 = 0^\circ$) with characteristic width of $y_w = 5^\circ$, $10^\circ$, $15^\circ$, and $20^\circ$ (Fig. 2.6a). For all cases we use a constant WES feedback parameter of magnitude 13.5 $J m^{-3}$, chosen because it is the average of $\alpha(y)$ in boreal spring between $30^\circ$S and $30^\circ$N. Optimal structures obtained in this system are fairly insensitive to similar WES feedback structures or amplitudes. The growth rates are somewhat more sensitive, but the hierarchy of growth rates for the different $K_q$ profiles remains unchanged, and so do the main conclusions. We note that for all symmetric mean states here the system is linearly stable and the least stable eigenvector of the system is symmetric.

Growth of the leading symmetric and anti-symmetric optimal structures under the different coupling regions (Fig. 2.6a) are shown in Fig. 2.6b and c, respectively. A general increase of the growth is observed for both symmetric and anti-symmetric leading optimals as the coupling region gets wider. The rate of increase of the growth as function of $K_q$ width is greater for anti-symmetric optimals compared to symmetric ones, indicating that anti-symmetric modes preferentially grow via non-normal processes. This is consistent with results from Xie 1996 that shows that the coupling of the system is proportional to the coupling window width squared for antisymmetric modes. For these model parameters the two narrowest $K_q$ configurations produce a symmetric leading optimal, and the two widest $K_q$ configurations produce an anti-symmetric leading optimal. The importance of the symmetric optimals for a narrow symmetric coupling window is somewhat at odds with results.
of Okajima 2003, and highlights the role of the coupling window specification. We have checked that the first optimal remains symmetric under the $5^\circ K_q$ window for different zonal wavenumbers covering at least $k = [0\ 20^{\circ}\pi]_{120^\circ}$ (not shown).

Comparing Fig. 2.4e to Fig. 2.5c, it is observed that the leading optimal condition achieves more growth in boreal fall compared to spring. Is this result part of a pattern in which optimal conditions transiently grow more for asymmetric mean states? To investigate this we calculate the transient growth under different $K_q(y)$ central locations. We use the $y_w = 5^\circ$ width function (Fig. 2.6 solid line), but vary $y_0$ as $0^\circ, 4^\circ, 8^\circ, 12^\circ, 16^\circ$ and $20^\circ$ for each different case. The same constant WES feedback profile used in the previous section is utilized here. The system is linearly stable under all $K_q$ configurations considered. We have restricted this part of the study to optimals with a large anti-symmetric component, as they are the ones associated with motions that mostly vary in the north-south axis. For $K_q$ centered at $0^\circ$ we show the first anti-symmetric optimal growth. If we include the symmetric mode, the conclusions derived from this section do not change. Figure 2.7 shows the transient growth as a function of $\tau$ for the different configurations. There is an increase in the maximum growth of optimals as a function of $y_0$, up to a peak that occurs when $y_0 = 13^\circ$ ($y_0 = 13^\circ$ case is not shown). Beyond $y_0 = 13^\circ$ the maximum transient growth starts to diminish, and starts occurring earlier in the system evolution.

2.3.3 The mean state as a "mode selector"

The different optimal responses for different symmetric mean states (Fig. 2.6a) may be understood in terms of the spatial structure of heating of stationary forced atmospheric Rossby and Kelvin waves, which depends on the product $K_q(y)T$. In this case, the mean state acts as a mode selector for the atmospheric contribution. To explore the role of ITCZ width and location on the forcing of atmospheric free modes, we calculate the norm of
$K_q(y)D_n(y)$ where $D_n(y)$ is the parabolic cylinder function of order $n$ (Gill 1980). This norm roughly indicates the amplitude of excitation of Kelvin and Rossby waves that would be generated by temperature anomalies with structure $D_n(y)$ for a given ITCZ structure [note that the norm of $D_n(y)$ is equal to 1]. We plot the norm of $K_q(y)D_n(y)$ for the first three parabolic cylinder functions as a function of $y_w$ in Fig. 2.8a and $y_0$ in Fig. 2.8b.

The transition from the symmetric mode for narrow ITCZ structures to an anti-symmetric mode for broader ITCZ structures is examined as a function of ITCZ width (all equatorially centered) in Fig. 2.8a. When $K_q(y)$ is confined to low latitudes, the only region where surface temperature anomalies can excite a significant atmospheric response is confined to the equator. In the case of the $5^\circ$ and $10^\circ$ width ITCZ cases, only the Kelvin wave and gravest symmetric Rossby mode are effectively excited by surface temperature variations. For wider $K_q(y)$ higher order Rossby modes are activated via the product of $T(y)$ with $K_q(y)$, including anti-symmetric ones. For reference, the $n = 1$ parabolic cylinder function, which is the heating structure that excites the lowest anti-symmetric Rossby wave, has a maximum at one Rossby radius away from the equator ($\sqrt{c/\beta}$), or about $10^\circ$ of latitude for the parameters used in this study. The symmetric initial condition will excite a damped equatorial Kelvin wave, which interferes with growth in this model through the WES feedback [the Kelvin wave produces easterly anomalies over positive temperature anomalies; see, e.g. Gill (1980)]. Hence, for wide coupling regions the anti-symmetric modes dominate.

Similar arguments explain why the anti-symmetric optimal is preferentially excited for ITCZ structures that are centered off the equator. The norm of $D_n(y)K_q(y)$ is plotted as a function of ITCZ location in Fig. 2.8b for $y_w = 5^\circ$. Heating is maximized for the leading symmetric parabolic cylinder function when the ITCZ is symmetric about the equator, and decreases as the ITCZ shifts poleward. Still, the damped Kelvin wave response for the leading symmetric parabolic cylinder function reduces growth through the WES feedback.
Heating for the second ($n = 1$, anti-symmetric) parabolic cylinder function maximizes when the ITCZ is centered at about 12°, near the location of maximum growth in Fig. 2.6. As the ITCZ moves poleward the effective coupling region from the ocean to the atmosphere moves with it. Vimont (2010) show that for larger values of the Coriolis parameter the atmospheric geopotential response to heating anomalies becomes more in-phase with the surface temperature, which reduces the effectiveness of the WES feedback in generating transient growth. Hence, growth is maximized for anti-symmetric structures that originate in the subtropics.

Comparing the growth and structures of the optimal responses under different symmetric coupling regions is not just a theoretical exercise. As previously mentioned, in this work the ocean is coupled to the atmosphere through SST induced deep heating anomalies. To a first approximation this Gill-like atmospheric model (Gill 1980) can be transformed into a Lindzen and Nigam (LN) (Lindzen and Nigam 1987) type model (Neelin 1989) where SST influences the boundary layer pressure gradients that directly drive the boundary layer winds. The equatorial radius of deformation ($L_D$) is typically smaller in the LN formulation (Neelin 1989; Vimont 2010). This implies that the same coupling region width (in physical units), will look wider in a LN type model, further implying different symmetry and growth rates for the leading optimals depending on which kind of model is considered. This is verified using the Vimont 2010 version of a LN-type model, where the convective coupling window affects boundary layer mass ventilation by deep convection. In this case the leading optimal structures remain equatorially antisymmetric for all the coupling windows considered in figure 2.6a (see fig. 1 Supplementary Material).
2.4 Conclusion

The role of the mean state in growth and structure of tropical meridional mode variability was examined using the simple linear coupled model of Vimont (2010). For an equatorial mean state that is largely symmetric (representing boreal spring in the Atlantic) growth is dominated by a zonally propagating eigenmode that does not resemble observed meridional mode variations. Transient growth of meridional-mode-like structures is larger for boreal summer and fall conditions, and the spatial structure of variability is more consistent with observed meridional mode variability.

The role of hemispheric asymmetry in optimal initial conditions that produce transient growth was examined through developing a northern or southern hemisphere initial norm under which to evaluate growth. Leading optimal conditions grow more when they start in the northern hemisphere compared to southern hemisphere during boreal spring, and SH constrained initial conditions do not grow at all during boreal fall. This is consistent with existing research that finds an important role for northern hemisphere compared to southern hemisphere extra tropical forcing in meridional mode variability. We caution, though, that these optimals were calculated by assuming that SST influences the atmosphere through its impact on deep convection. A different result may result in a model where SST is coupled to the atmosphere through boundary layer SST induced pressure variations, an effect that we have not considered in depth in this work.

Optimal structures and growth were examined for different mean states, with a focus on ITCZ width and location as the WES feedback is relatively insensitive to seasonal variations. For an equatorially symmetric ITCZ optimal structures were either purely symmetric or anti-symmetric about the equator, and evolve conserving their initial symmetry. Internal dynamics do not mix the symmetries for this idealized case. This does not preclude external
shocks (like asymmetric external atmospheric forcing) from mixing the symmetry of the optimals. For narrow ITCZ structures equatorially symmetric optimals dominate the growth, while anti-symmetric optimal structures dominate growth for wider ITCZ structures. This is traced to the increase in importance of the anti-symmetric atmospheric Rossby wave response as the (symmetric) effective coupling region gets broader.

Growth and optimal structures were also examined as a function of ITCZ location, which relates to seasonal variations in the ITCZ. In that case, the optimal response activity was mainly contained in the same hemisphere as the ITCZ, as there is not enough convective support for an atmospheric response that would feedback and grow outside of the ITCZ region. As a consequence of this, optimals under an asymmetric ITCZ are neither symmetric or anti-symmetric. Interestingly, optimal initial conditions experience maximum growth when the ITCZ is located off of the equator, which is counter-intuitive considering that maximum meridional mode variance is found in boreal spring (Chiang and Vimont 2004). This could imply that meridional mode variability during boreal summer and fall is more transiently maintained by internal ocean-atmosphere interactions, whereas in spring the thermodynamic coupling might have a secondary role to direct atmospheric forcing. More studies are needed to assess the relative roles of internal thermodynamically coupled ocean-atmosphere interactions and direct atmospheric forcing on meridional mode variability during different seasons.

There are numerous caveats to the findings in this study. First, the atmospheric model is overly simplistic though results should be applicable to a Lindzen and Nigam (1987) or Battisti et al. (1999) style model of the boundary layer response to surface temperature anomalies. Second, the ocean model is clearly too simplistic as lateral Ekman fluxes, upwelling, ocean dynamics as well as low-cloud feedbacks will play a role in surface temperature variations in nature. Third, in reality the ITCZ and WES feedback are co-determined by the
dynamics of the Hadley and Walker circulations. No attempt was made herein to ensure a physically consistent mean state, aside from deriving the specific mean states for the Atlantic from observations.

Despite these caveats, the study does provide physical insight into meridional mode behavior under structural variations in the mean state. Specifically, it highlights the importance of the mean state as a “mode selector” of thermodynamically coupled ocean-atmosphere variability: i.e. through preferentially enabling ocean-to-atmosphere coupling for specific atmospheric modes, the mean state can play an important role in determining whether tropical thermodynamically coupled variability is dominated by equatorially symmetric, antisymmetric, or asymmetric structures. Results also identify the convective coupling provided by the ITCZ as a key player in meridional mode dynamics.

Acknowledgments.

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Table 2.1: Model parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_o )</td>
<td>( 10^3 \text{ kg m}^{-3} )</td>
</tr>
<tr>
<td>( c_o )</td>
<td>( 4.2 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1} )</td>
</tr>
<tr>
<td>( H_o )</td>
<td>( 40 \text{ m} )</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>( 2 \text{ d}^{-1} )</td>
</tr>
<tr>
<td>( \epsilon_T )</td>
<td>( 120 \text{ d}^{-1} )</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>( 10^3 \text{ m}^2 \text{ s}^{-1} )</td>
</tr>
</tbody>
</table>
Figure 2.1: a) Feb-Mar-Apr average of $P - E$ over the Atlantic for 1981-2010. Contour units: $15 \times 10^{-6} \frac{Kg}{m^2s}$. b) Zonal average of part a ($10^{-6} \frac{Kg}{m^2s}$). c) $K_q(y)$ ($10^{-3} \frac{m^2}{s^3K}$). d) Same as a but for Aug-Sep-Oct. e) Same as b but for Aug-Sep-Oct. f) Same as c, but for Aug-Sep-Oct. Monthly 1981-2010 precipitation and latent heat data from the ERA Interim reanalysis (Dee et al. 2011) was used.
Figure 2.2: Zonal mean of the WES parameter over the Atlantic for 1981-2010. Feb-Mar-Apr average (solid). Aug-Sep-Oct average (dashed). The unit of the horizontal axis is $\frac{J}{m^3}$. Monthly 1981-2010 latent heat and wind data from the ERA Interim reanalysis (Dee et al. 2011) was used.
Figure 2.3: Optimal initial (a and c) and final (b and d) structures calculated under the same mean state as used in Vimont (2010) for 140d under the full (unconstrained) initial norm kernel and SST final norm kernel (a and b) and for 100d under the NH initial norm kernel with full (L2) final norm kernel (c and d). SST is represented by gray contours, low level geopotential height by black contours and low level wind fields by the arrows. Contours are shown at increments of 0.3 times the maximum value of the field, solid contours denote positive values and dashed contours negative ones. Units on the x and y axes have been scaled by the zonal wavelength ($\frac{2\pi}{k} = 120^\circ$) and the equatorial radius of deformation ($L_y \sim 10^\circ$ for the parameters of this model). e: Growth (under the L2 norm) as a function of days of the: (i) optimal calculated under the full L2 norm (Vimont 2010) (black line behind grey dashed line); (ii) optimal calculated under the SST norm, shown in a and b (grey dashed line); (iii) optimal calculated under the NH initial norm, shown in c and d (black dash-dot).
Figure 2.4: Optimal initial (a and c) and final (b and d) structures calculated under the Feb-Mar-Apr mean state for 140d under the full (unconstrained) initial norm kernel (a and b) and for 140d under the NH initial norm kernel (c and d). SST is represented by grey contours, low level geopotential height by black contours and low level wind fields by the arrows. Contours are shown at increments of 0.3 times the maximum value of the respective initial condition fields, solid contours denote positive values and dashed contours negative ones. The lengths of the velocity arrows are drawn relative to the maximum wind speed of the respective initial conditions. Units on the x and y axes have been scaled by the zonal wavelength ($\frac{2\pi}{k} = 120^o$) and the equatorial radius of deformation ($L_y \sim 10^o$ for the parameters of this model). Panel e shows the growth $\lambda_i$ as a function of days for: (i) the unconstrained optimal initial condition (black solid); (ii) optimal NH initial condition (grey dashed), (iii) optimal SH initial condition (black dotted); and (iv) a 120 day decay curve (grey dash-dot).
Figure 2.5: a and b: 110 Days Initial and Final Optimal structures for Aug-Sep-Oct mean states when the initial condition is unconstrained. SST is represented by grey contours, low level geopotential height by black contours and low level wind fields by the arrows. Contours are shown at increments of 0.3 times the maximum value of the initial condition fields, solid contours denote positive values and dashed contours negative ones. The lengths of the velocity arrows are drawn relative to the maximum wind speed of the initial condition. Units on the x and y axes have been scaled by the zonal wavelength ($\frac{2\pi}{k} = 120^\circ$) and the equatorial radius of deformation ($L_y \sim 10^\circ$ for the parameters of this model). Panel e shows the growth $\lambda_t$ as a function of days for: (i) the unconstrained optimal initial condition (black solid; note this is behind the grey dashed line); (ii) optimal NH initial condition (grey dashed), (iii) optimal SH initial condition (black dotted); and (iv) a 120 day decay curve (grey dash-dot).
Figure 2.6: **a**: Different $K_q$ curves. The peak is $2.25 \frac{m^2}{s^3 K}$. The curves are described in terms of their characteristic width: i) $5^\circ$ solid, ii) $10^\circ$ dashed line, iii) $15^\circ$ dash-dot line, and iv) $20^\circ$ dotted line. The vertical axis is measured in degrees. **b**: Least stable symmetric optimal growth as function of $K_q$ characteristic width $y_w$. **c**: Least stable anti-symmetric optimal growth as a function of $K_q$ of $y_w$. 
Figure 2.7: Growth of the least stable asymmetric mode under mean state given by the shape and magnitude of solid line in figure 2.6a, but centered at \( y_0 = (i) \) 0° (solid), (ii) 4° (dashed), (iii) 8° (dash-dot), (iv) 12° (dotted), (v) 16° (circle), and (vi) 20° (cross).
Figure 2.8: **a**: Norm of $K_q(y)D_n(y)$ as a function of $y_w$ (for $y_0 = 0^\circ$). **b**: Norm of $K_q(y)D_n(y)$ as a function of $y_0$ (for $y_w = 10^\circ$). This shows that **a** the effective coupling increases as the ITCZ width increases, and that **b** the ITCZ location acts as a mode selector as it moves poleward.
Supplemental Material

We consider the Vimont 2010 version of the Battisti-Sarachik-Hirst (1999) reduced-gravity model of the well mixed tropical boundary layer

\[
\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \\ \phi \\ T \end{pmatrix} = \begin{pmatrix} -\varepsilon & \beta y & -g'ik & \kappa \Gamma_{RG}ik \\ -\beta y & -\varepsilon & -g'\frac{\partial}{\partial y} & \kappa \Gamma_{RG} \frac{\partial}{\partial y} \\ -H_b ik & -H_b \frac{\partial}{\partial y} & -\epsilon_M(y) & 0 \\ \alpha(y) & 0 & 0 & \varepsilon_T + \gamma \nabla^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ \phi \\ T \end{pmatrix} \tag{2.11}
\]

where \( h \) denotes anomalous boundary layer height from the climatological value \( H_b \). In this model virtual potential temperature is proportional to surface temperature (\( \sim \kappa T \)) and is well mixed in the boundary layer. The potential temperature anomalies are responsible for hydrostatic pressure perturbations anomalies acted upon by reduced gravity. The boundary layer mass is vented out by deep convection at the rate \( \epsilon_m(y) \) shown in figure R1a. The remaining parameter values are the following: \( \kappa = 1 \), \( \Gamma_{RG} = 50 \frac{m^2}{s^3} \), \( H_b = 3000 m \), \( g' = 0.1 \frac{m}{s^2} \). Additional details of the model can be found in Vimont 2010.
Figure 2.9: a: Different ventilation timescales as a function of the different convective windows. b. Least stable symmetric optimal for the different ventilation timescales. c. Least stable antisymmetric optimal for the different ventilation timescales.
Chapter 3:

AN ANALYTICAL FRAMEWORK FOR UNDERSTANDING TROPICAL MERIDIONAL MODES

The content of this paper has been submitted to Journal of Climate as Martinez-Villalobos, C. and D. J. Vimont, 2016: An analytical framework for understanding tropical Meridional Modes. A pre-print version may be found at https://arxiv.org/abs/1607.07915

Abstract

A theoretical framework is developed for understanding the transient growth and propagation characteristics of thermodynamically coupled, meridional mode-like structures in the tropics. The model consists of a Gill-Matsuno type steady atmosphere under the longwave approximation coupled via a wind-evaporation-sea surface temperature (WES) feedback to a “slab” ocean model. When projected onto basis functions for the atmosphere the system simplifies to a non-normal set of equations that describes the evolution of individual sea surface temperature (SST) modes, with clean separation between symmetric and anti-symmetric modes. The following major findings result from analysis of the system: (i) a transient growth process exists whereby specific SST modes propagate toward lower order modes at the expense of the higher-order modes; (ii) the same dynamical mechanisms gov-
ern the evolution of symmetric and anti-symmetric SST modes except for the lowest-order wave number, where for symmetric structures the atmospheric Kelvin wave plays a critically different role in enhancing decay; and (iii) the WES feedback is positive for all modes (with a maximum for the most equatorially confined antisymmetric structure) except for the most equatorially confined symmetric mode where the Kelvin wave generates a negative WES feedback. Taken together, these findings explain why equatorially anti-symmetric “dipole”-like structures may dominate thermally coupled ocean / atmosphere variability in the tropics. The role of non-normality as well as the role of realistic mean states in meridional mode variability are discussed.
3.1 Introduction

Distinct patterns of low frequency tropical/subtropical ocean-atmosphere coupled variability in the Pacific and Atlantic occurs mainly via two distinct feedbacks: the so-called Bjerknes feedback [Bjerknes (1969); though the Bjerknes feedback is now understood to involve additional dynamical processes in the ocean (Trenberth et al. 1998)] and Wind-Evaporation-SST (WES) feedback (Xie and Philander 1994; Chang et al. 1997). Both of these feedbacks imply a mutual reinforcement between winds and sea surface temperature (SST), but via fundamentally different mechanisms. The Bjerknes feedback relies on ocean dynamical processes in linking wind anomalies to SST tendencies, and typically results in coupled modes that are largely equatorially symmetric such as the El Niño / Southern Oscillation (ENSO). Outside the equatorial zone but within the tropics surface heat fluxes, especially latent and short-wave, play an increasingly important role in SST variability. There, the WES feedback links surface winds to SST tendency through changes in evaporation rates: if positive SST induced wind anomalies are directed opposite to the mean wind direction, the resulting decrease in total wind speed is associated with a reduction in evaporation, and hence a positive SST tendency. The feedback loop is completed in both cases by SST affecting the winds via deep convection (Gill 1980) or boundary layer processes (Lindzen and Nigam 1987; Battisti et al. 1999).

The WES feedback has mainly been associated with maintaining equatorially antisymmetric coupled modes [meridional modes; Servain et al. (1999); Chiang and Vimont (2004)] by the following mechanism. Consider an anomalous equatorially antisymmetric SST dipole. The atmospheric response to such a dipole includes anomalous winds blowing from the cold hemisphere into the warm hemisphere. Due in partly to the Coriolis effect, these anomalous winds relax the mean easterly trades in the warm hemisphere and reinforce them in the cold
hemisphere. By wind modulation of evaporation the SST dipole is reinforced. This mechanism has been used to explain the maintenance of meridional modes of ocean-atmosphere covariability such as the Atlantic Meridional Mode (Moura and Shukla 1981; Chang et al. 1997), and the Pacific Meridional Mode (Chiang and Vimont 2004). Here, we show that interhemispheric anti-symmetry is not necessary for meridional mode evolution and growth.

The mechanisms of growth of meridional modes have been the subject of several theoretical studies. Some of these studies have focused on modal (normal) growth of the least stable eigenvector of a dynamical system that contains the thermodynamic coupling (Noguchi 1998; Xie 1999; Xie et al. 1999; Wang 2010), while others have investigated the collective transient (non-normal) growth due to positive interference between the individual modes (Vimont 2010; Martinez-Villalobos and Vimont 2016b). The propagation characteristics of this kind of modes has also been the subject of scholarly interest. Liu and Xie (1994) and Xie (1999) demonstrate that these coupled anomalies propagate westward and equatorward. The equatorward propagation occurs due to wind anomalies that are centered equatorward of SST anomalies, while the westward propagation occurs due to the westward phasing of the Rossby wave atmospheric response to the SST anomalies (Xie 1996).

Although the WES feedback is usually seen as responsible for equatorially antisymmetric variability, a few studies (Noguchi 1998; Xie 1999; Vimont 2010; Wang 2010; Martinez-Villalobos and Vimont 2016b) have shown that the WES feedback is also able to maintain equatorially symmetric modes. More recently Clement et al. (2011), using an ensemble of General Circulation Models with “slab” ocean models (i.e. no Bjerknes feedback included), showed the existence in the equatorial Pacific of ENSO-like ocean-atmosphere modes maintained only by thermodynamic fluxes. It is possible that such a mode exists in nature, but its manifestation is overshadowed by the Bjerknes feedback.

This paper aims to provide basic insight into the similarities and differences between large-
scale equatorially symmetric and antisymmetric (meridional mode-like) modes coupled by the WES feedback. Special attention will be given to analyzing mechanisms of propagation and growth for this two set of modes. We will refer to these variations as “meridional modes” irrespective of their equatorial symmetry properties. The name “meridional modes” is usually used to refer to variations that are equatorially antisymmetric (e.g. a meridional dipole), but its use here in a more general sense is justified (as it will be argued later) as growth and propagation of both equatorially symmetric and antisymmetric modes involves the same dynamics through the WES feedback.

The remainder of the paper is organized as follows. Section 2 describes the model and methods used. Section 3 describes and discusses the results. Finally section 4 will present the conclusions.

3.2 Model Description

The model used herein to investigate equatorially symmetric and antisymmetric variability consists of the atmospheric Gill-Matsuno model (Matsuno 1966; Gill 1980) coupled to a thermodynamic “slab” ocean. To simplify the calculations and facilitate the interpretation of the results we will adopt the “long wave approximation” (Gill and Clarke 1974; Gill 1980) in the atmospheric equations, though in Section 3.3.5 we will relax that assumption. For practical purposes the approximation consists of dropping the atmospheric damping in the $v$ equation so the zonal wind is in geostrophic balance. In non-dimensional form, this model
may be written:

\[
\begin{align*}
\epsilon u - yv &= -\frac{\partial \phi}{\partial x} \\
yu &= -\frac{\partial \phi}{\partial y} \\
\epsilon \phi + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= -K_q T \\
\frac{\partial T}{\partial t} + \epsilon_T T &= \alpha u,
\end{align*}
\]

where \( u \) and \( v \) denotes low-level zonal and meridional wind anomalies, \( \phi \) low level geopotential anomalies, and \( T \) SST anomalies. In this model the atmosphere is diagnostic and completely determined by the underlying SST. The SST affects the atmosphere through deep heating, with coupling parameter denoted as \( K_q \). The atmosphere affects the SSTs through changes in evaporation due to zonal wind anomalies, with parameter \( \alpha \) controlling the strength of that coupling. The parameter \( \alpha \) can be derived in a variety of ways: Czaja et al. (2002) and Vimont et al. (2009) derive \( \alpha \) using a Taylor expansion of the bulk latent heat flux formula around variations in the zonal wind \( u \). Notice that in the presence of mean easterlies \( (\alpha > 0) \), a positive \( u \) anomaly implies a reduction in wind speed and hence evaporation, and thus induces a positive SST tendency. We take both \( K_q \) and \( \alpha \) parameters to be constant, though the model could be easily expanded to include a spatial dependence of \( K_q \) or \( \alpha \) (this is discussed further in Section 3.4.2). Both SST and atmospheric parameters are linearly damped by terms \( \epsilon_T \) and \( \epsilon \) respectively. The effective coupling of the system is given by \( K_q \alpha \) (Martinez-Villalobos and Vimont 2016b). The parameter values are shown in Table 3.1. As we are interested in large scale variability, the Gill-Matsuno atmospheric model is appropriate, though Vimont (2010) investigates a similar model framework using both the Gill-Matsuno (no long wave approximation used) and Battisti et al. (1999) reduced-gravity boundary layer models. Remaining details of the model are described in Vimont (2010) and
Martinez-Villalobos and Vimont (2016b).

Note that in (3.1) the SST tendency is coupled to the atmosphere only through zonal wind anomalies $u$, and that the steady atmospheric response can in turn be expressed entirely in terms of temperature. We express the atmosphere in terms of the SST alone by decomposing the SST in the meridional direction using parabolic cylinder functions $\psi_m(y)$ (see appendix C):

$$T(t, y, x) = \sum_{m=0}^{\infty} T_m(t) \psi_m(y) \exp(ikx),$$ (3.2)

where $k$ is the zonal wavenumber considered. A particular parabolic cylinder function $\psi_m$ has $m$ zeros, has the symmetry of $m$ ($m$ even corresponds to equatorially symmetric structures, and $m$ odd to equatorially antisymmetric structures), and peaks further from the equator as $m$ increases (see figure C.1). So, a low latitude SST signal is dominated by low $m$ modes, while a high latitude signal is dominated by high $m$ modes.

The atmosphere may be written in terms of the SSTs (see appendix D) as:

$$u(t, y, x) = \frac{1}{2} (q_0(t) \psi_0(y) + \sum_{m=1}^{\infty} q_{m+1}(t)(\psi_{m+1}(y) - \sqrt{\frac{m+1}{m}} \psi_{m-1}(y))) \exp(ikx)$$

$$v(t, y, x) = \sum_{m=0}^{\infty} \sqrt{\frac{1}{2(m+1)}} (K_q T_{m+1}(t) + (\epsilon + ik) q_{m+1}(t)) \psi_m(y) \exp(ikx)$$

$$\phi(t, y, x) = \frac{1}{2} (q_0(t) \psi_0(y) + \sum_{m=1}^{\infty} q_{m+1}(t)(\psi_{m+1}(y) + \sqrt{\frac{m+1}{m}} \psi_{m-1}(y))) \exp(ikx)$$ (3.3)

where $q_0$ and $q_{m+1}$ are the amplitudes of the atmospheric Kelvin wave and $m$'th Rossby wave respectively. These amplitudes are determined by the SST as:

$$q_0 = -K_q \frac{T_0}{\epsilon + ik}$$

$$q_{m+1} = K_q \left( \frac{\sqrt{m(m+1)} T_{m-1} + mT_{m+1}}{-(2m+1)\epsilon + ik} \right)$$ (3.4)
Note that the denominator in (3.4) includes a term related to the inverse of the non-dimensional group velocity for non-dispersive (due to the long wave approximation) Kelvin (+1) or Rossby (−(2m + 1)) waves (Gill 1980). These terms will be prove useful in tracking the role of Rossby or Kelvin waves in the system evolution.

Using (3.3) and (3.4) in the SST equation for (3.1) we may write an equation for a particular $T_m$ just in terms of SST modes.

$$\frac{\partial T_m}{\partial t} = \frac{1}{2} K_q \alpha \left[ \sqrt{(m-1)m} T_{m-2} + \left( \frac{m-1}{(2m-1)\epsilon + ik} - \frac{m+2}{(2m+3)\epsilon + ik} \right) T_m - \frac{(m+1)(m+2)}{(2m+3)\epsilon + ik} T_{m+2} \right] \epsilon T_m. \quad (3.5)$$

Also an equation for the squared amplitude of a particular mode can be derived (see appendix E equation E.3). We defer the analysis of the terms in these equations to section 3.3.2. For now, it is important to notice that the effect of the WES feedback in the evolution of SST anomalies is enclosed in the $K_q \alpha$ term, and the SST damping is just the last term. Also, (3.4) and (3.5) shows explicitly that $T_m$ is affected by Rossby wave $m-1$ (or Kelvin wave for $T_0$) and $m+1$ (note the $2m-1$ or $2m+3$ terms in the denominator, which relates to the Rossby wave group velocity as discussed above), and that symmetric and antisymmetric modes do not interact. This is a consequence of $K_q$ and $\alpha$ being symmetric. Vimont (2010), and Martinez-Villalobos and Vimont (2016b) have studied how meridional variations in the coupling affect thermodynamically coupled variability.

Because symmetric and antisymmetric modes do not interact, we can analyze them separately. Figure 3.1 shows an example of equatorially symmetric and antisymmetric structures and how they evolve in time. Also plotted is their growth (measured as the basin integrated SST variance, see equation E.5) through the WES feedback. These structures correspond to the leading equatorially symmetric and antisymmetric initial conditions that maximize
growth (Farrell and Ioannou 1996; Vimont 2010) at 180 days of evolution under parameters shown in Table 3.1 (Appendix F includes a discussion of growth optimization). The important point for the present discussion is to point out that both symmetric and antisymmetric patterns grow over time (although the antisymmetric pattern grows more), SST structures propagate equatorward and westward for both patterns, and there are processes responsible for the propagation and growth that seemingly work in a similar way for both patterns, namely positive (negative) zonal wind anomalies over positive (negative) SST anomalies.

3.3 Results and Discussion

3.3.1 Low and high SST modes

In this section we will analyze the evolution of a particular $T_m$ mode from (3.5) for low and high $m$ SST modes. We show that a particular mode will propagate “outward” to higher and lower order modes, while retaining symmetry. This continues until the lowest-order mode, which excites either the Kelvin wave (symmetric) or mixed Rossby-Gravity wave (anti-symmetric), with fundamental consequences on the growth rate of symmetric or anti-symmetric structures.

From (3.5) we notice that for $m \geq 2$ both symmetric ($m$ even) and antisymmetric ($m$ odd) SST modes evolve in a qualitatively similar way, via excitation of the $(m+1)$ and $(m-1)$ Rossby waves. Note that the $T_{m+2}$ structure excites the $(m+1)$ and $(m+3)$ Rossby waves (see 3.4); the former includes $u$ anomalies that project onto $\psi_m$ [last term multiplying $K_q \alpha$ in 3.5]. Similarly, the $T_{m-2}$ structure excites the $(m-3)$ and $(m-1)$ Rossby waves (see 3.4); the latter includes $u$ anomalies that project onto $\psi_m$ [first term multiplying $K_q \alpha$ in 3.5]. And finally, the $T_m$ structure itself excites both $(m-1)$ and $(m+1)$ Rossby waves, which both include $u$ anomalies that project onto $\psi_m$ [middle term multiplying $K_q \alpha$ in 3.5]. This demonstrates
that for $m \geq 2$ a particular symmetric or antisymmetric mode will propagate “outward”
toward the next higher and next lower symmetric or antisymmetric mode, respectively) 
through Rossby wave excitation. In physical space, the propagation toward different modes 
represents a meridional propagation toward a higher latitude variation (for $m$ increasing) or 
toward more equatorially confined variability (for $m$ decreasing).

For $m = 0$ and $m = 1$ the situation is different. In this case the evolution of both $T_0$ and $T_1$ modes are dictated by qualitatively different equations,

$$\frac{\partial T_0}{\partial t} = \frac{1}{2} K_\alpha \left[ \left( \frac{-1}{3\epsilon} + \frac{2}{-3\epsilon + ik} \right) T_0 - \sqrt{2} \left( \frac{-3}{-3\epsilon + ik} T_2 \right) \right] - \epsilon T_0$$

$$\frac{\partial T_1}{\partial t} = \frac{1}{2} K_\alpha \left[ \left( \frac{-3}{5\epsilon + ik} T_1 - \frac{\sqrt{6}}{-5\epsilon + ik} T_3 \right) \right] - \epsilon T_1.$$  \hspace{1cm} (3.6)

The evolution of the $T_0$ term is affected by both atmospheric Kelvin wave and the first symmetric Rossby wave, whereas the $T_1$ term is affected by just the first antisymmetric Rossby wave [the next-lower anti-symmetric mode is the mixed Rossby-Gravity wave, for which $u$ anomalies under the long-wave approximation are strictly zero as shown in (3.4) (i.e. $q_1 = 0$)].

Taken together, the full evolution from $m \geq 2$ toward $m = 0, 1$ modes illustrates two important characteristics of thermodynamically coupled variability. First, for $m \geq 2$ the dynamics of both equatorially symmetric and antisymmetric SST patterns is essentially the same. Second, the dynamics of the system only differ for the lowest order modes $m = 0, 1$, where the Kelvin (for $m = 0$) and mixed Rossby-Gravity (for $m = 1$) modes have fundamentally different effects. Note from Fig. 3.1 that the total growth for the symmetric and antisymmetric patterns is quite different. We will demonstrate below that the reason for this difference lies in the atmospheric Kelvin wave.
3.3.2 Analysis of terms in SST equation

In order to analyze (3.5) in the parameter space considered herein we rewrite it in terms of the ratio between the wavenumber and atmospheric damping ($\nu$), and the ratio between the coupling and damping terms ($\sigma$ the stability parameter):

$$\nu = \frac{k}{\epsilon},$$

$$\sigma = \frac{K_q \alpha}{\epsilon_T \epsilon},$$

(3.7)

as

$$\frac{\partial T_m}{\partial t} = \frac{1}{2} \epsilon_T (-h(m - 1, \nu)\sigma T_{m-2} + f(m, \sigma, \nu) T_m + h(m + 1, \nu)\sigma T_{m+2}),$$

(3.8)

where the functions introduced are defined as:

$$h(m, \nu) = \frac{\sqrt{m(m + 1)}}{(2m + 1) - i\nu}$$

$$f(m, \sigma, \nu) = g(m, \nu)\sigma - 2$$

$$g(m, \nu) = \frac{(2m + 1) - 3i\nu}{(2m - 1)(2m + 3) - 2i\nu(2m + 1) - \nu^2}.$$  (3.9)

The function $h(m, \nu)$, the exchange function, encodes part of the influence of Rossby wave $m$ [the atmospheric Kelvin wave does not participate in the exchange as $h(-1, \nu) = 0$] via connecting $T_{m-2}$ and $T_{m+2}$ to $T_m$, and its real part is always positive. Notice that the sign of $h(m, \nu)$ in (3.8) shows that both $T_{m-2}$ and $T_{m+2}$ terms affect $T_m$ evolution in different directions (opposite polarity). This will prove important in explaining the propagation characteristics and transient growth of the modes.

Function $f(m, \sigma, \nu)$ contains the effect of $T_m$ on itself. Here, the function $g(m, \nu)\sigma$ contains part of the WES feedback influence on growth (if $g > 0$ it contributes to $T_m$ growth,
and if \( g < 0 \) it contributes to \( T_m \) decay) and the \(-2\) term corresponds to the SST damping. Notice that the instantaneous growth or decay of a single \( T_m \) mode depends on the balance between the WES feedback \([g(m, \nu)\sigma]\) and the SST damping \((-2\sigma)\), where here we distinguish the WES feedback as including \textit{only} the wind-induced change in surface latent heat flux.

### 3.3.3 Long zonal wavelength limit

As a starting point we will study the long zonal wavelength \((k \to 0)\) limit, i.e. \( \nu = 0 \). This limit will prove useful in understanding the more general variability for long but finite zonal wavelengths. In this case (3.5) (also 3.8) is reduced to

\[
\frac{\partial T_m}{\partial t} = \frac{1}{2} \epsilon_T (-h(m-1,0)\sigma T_{m-2} + f(m,\sigma,0)T_m + h(m+1,0)\sigma T_{m+2}).
\]  

(3.10)

Functions \( h \) and \( f \) in this limit are purely real, as in this limit atmospheric waves are zonally in phase with the SST:

\[
h(m,0) = \sqrt{\frac{m(m+1)}{2m+1}}, \quad f(m,\sigma,0) = g(m,0)\sigma - 2 = \frac{(2m+1)\sigma}{(2m-1)(2m+3)} - 2.
\]  

(3.11)

In this case, the equation for a particular mode growth (see appendix E equation E.3) is vastly simplified to

\[
\frac{\partial |T_m|^2}{\partial t} = \epsilon_T (-h(m-1,0)\sigma T_{m-2}T_m + f(m,\sigma,0)|T_m|^2 + h(m+1,0)\sigma T_{m+2}T_m).
\]  

(3.12)

Figure 3.2a shows \( h(m,0) \) as a function of mode \( m \). This shows that \( h \) is a very flat function of \( m \) for \( m > 1 \) (\( h(m-1,0) = 0 \) for \( m = 0, 1 \)). The difference in value between \( h(m+1,0) \) and \( h(m-1,0) \) is positive and small, and gets progressively smaller as \( m \) increases (\( \frac{dh(m,0)}{dm} \sim \)).
This implies that if \( T_{m-2} \) has the same magnitude as \( T_{m+2} \) their effect on \( T_m \) growth roughly cancels out, though the effect of \( T_{m+2} \) slightly dominates. This dominant influence of the mode \((m+2)\) on the lower-order \( m \) helps explain the equatorward propagation (i.e. propagation towards lower \( m \)) seen in figure 3.1.

Next, consider the growth equations for modes \( m-2 \) and \( m+2 \)

\[
\frac{\partial |T_{m-2}|^2}{\partial t} = \epsilon_T (-h(m-3,0)\sigma T_{m-4}T_m + f(m-2,\sigma,0)|T_{m-2}|^2 + h(m-1,0)\sigma T_m T_{m-2}).
\]

(3.13)

\[
\frac{\partial |T_{m+2}|^2}{\partial t} = \epsilon_T (-h(m+1,0)\sigma T_m T_{m+2} + f(m+2,\sigma,0)|T_{m+2}|^2 + h(m+3,0)\sigma T_{m+4}T_{m+2}).
\]

(3.14)

We notice that the third term in equation 3.13 is exactly equal to the first term in equation 3.12, but with opposite sign; similarly with the first term of equation 3.14 and third term in equation 3.12. Without loss of generality also consider that \( T_{m-2}, T_m \) and \( T_{m+2} \) all start positive. This shows that a positive \( T_{m+2} \) will grow \( T_m \) amplitude, while at the same time \( T_m \) will decrease \( T_{m+2} \) amplitude. Then \( T_m \) will increase \( T_{m-2} \) amplitude while \( T_{m-2} \) will damp \( T_m \). As a consequence of this process \( T_{m+2} \) grows \( T_m \) that then grows \( T_{m-2} \). This, at the core, is the mechanism that explains equatorward propagation (propagation towards lower order modes) of thermodynamically coupled variability.

The \( f(m,\sigma,0) \) function in (3.10) and (3.11) is key in the prospect of long term growth of a particular mode. This function depends on the particular mode \( m \) and the ratio between coupling and damping, the stability parameter \( \sigma \). A particular mode \( m \) will grow (at least for some time) if \( f(m,\sigma,0) > 0 \), otherwise it decays. Figure 2b shows how \( f(m,\sigma,0) \) varies with \( m \) (for the first 10 \( m \) modes), for different values of the stability parameter. Here \( \sigma_0 = 4.83 \) using the standard values shown in table 3.1. For the \( m \) values of interest \( g(m,0) \) has a pole
at \( m = \frac{1}{2} \) implying, as shown in figure 3.2 for 3 different values of \( \sigma \), a very different behavior for modes \( m = 0 \), and \( m = 1 \). In this limit \( g(m, 0) \) peaks at \( m = 1 \), is positive for \( m \geq 1 \), and is negative just when \( m = 0 \). This shows that the WES feedback is positive (it works towards amplifying an initial \( T_m \) anomaly) for all \( m \) modes, except \( m = 0 \). The function \( g(m, 0) \) gets monotonically smaller as \( m \) increases, implying that the WES feedback gets progressively less effective as we move to higher latitudes, consistent with Vimont (2010).

For \( m \geq 1 \) there is a tension in function \( f(m, \sigma, 0) \) between a positive WES feedback represented by the \( g(m, 0)\sigma \) term, that acts to enhance \( |T_m|^2 \) growth, and the SST damping represented by the \(-2\) term, that acts to damp SST. When \( g(m, 0)\sigma \) is greater than 2 there is a prospect for transient growth. In the cases shown in figure 2b, \( f(m, \frac{1}{2}\sigma_0, 0) < 0 \) for all \( m \)'s so the system simply decays, though at a rate slower than \( \epsilon^{-1} \). For \( \sigma = \sigma_0 \), \( f(m, \sigma_0, 0) > 0 \) just for \( m = 1 \) so the system will grow, at least for some time, just when the SSTs project onto \( \psi_1(y) \). Finally, for \( \sigma = 2\sigma_0 \), \( f(m, 2\sigma_0, 0) > 0 \) for \( m = 1, 2 \), so the SSTs will grow if they project onto \( \psi_1 \) and \( \psi_2 \), but still the greatest growth will be achieved when they project onto \( \psi_1 \). For \( m = 0 \) \( g(m, 0)\sigma < 0 \), so the WES feedback turns into a negative feedback, and the system is additionally damped for that mode.

The reason why \( T_0 \) is damped by, and \( T_1 \) grows through the WES feedback lies on the differences between atmospheric Kelvin and Rossby waves. Consider an equatorially symmetric zonally homogeneous positive SST anomaly confined close to the equator. The atmospheric response includes (see equation 3.4 for \( q_0 \) and \( q_2 \)) a positive zonal wind anomaly \( u_R \) associated with the first symmetric Rossby wave on top of the anomaly (recall, there is zero zonal phase difference between SST and the atmosphere in the \( \nu = 0 \) case), and a negative zonal wind anomaly \( u_K \) associated with the Kelvin wave. Since the mean zonal winds are negative along the equator (\( \alpha > 0 \)), the Rossby response acts to decrease evaporation further increasing the positive SST anomaly, whereas the Kelvin wave acts to increase evaporation.
damping the SST anomaly.

This result implies a very different situation between symmetric and antisymmetric SST patterns with respect to growth. A subtropical *antisymmetric* pattern will propagate equatorward (towards lower $m$) when it will finally project onto $\psi_1$. At that point the system will grow while at the same time causing a $T_3$ tendency in the opposite direction (see equation 3.12 for $m = 1$). As $T_3$ grows in the opposite direction, it ultimately counteracts the growth of $T_1$ causing the entire system to decay back to zero. On the other hand, a subtropical equatorially *symmetric* pattern will also propagate towards lower $m$, finally projecting onto $\psi_0$ where it will be additionally damped by the WES feedback due to the atmospheric Kelvin wave effect (see equation 3.6 with $k = 0$).

To illustrate how the Kelvin wave affects growth, we will consider the $\sigma = \sigma_0$ case from here on. In this case just the $m = 1$ mode has prospective growth. For the rest of the paper we will use the first 10 $T_m$ modes in our calculations (5 symmetric and 5 antisymmetric). We have tested that results do not change qualitatively by increasing the number of modes retained. Figure 3 shows how this asymmetry in the evolution of equatorially symmetric and antisymmetric SST anomalies in this limit plays out. To make things easier to interpret, the system will be initialized as $T(0, y) = \psi_4(y)$ in the symmetric case, and $T(0, y) = \psi_5(y)$ for the antisymmetric case. In accordance with (3.10) both patterns start very similarly. The symmetric (anti-symmetric) pattern generates a positive $T_2$ ($T_3$) response, and a similar but albeit smaller, negative $T_6$ ($T_7$) response, while being damped by these modes in the process. Both positive signals continue propagating equatorward, losing energy in the process until the antisymmetric SST pattern projects onto $\psi_1$, where the system grows for some time, and the symmetric SST pattern project onto $\psi_0$, where the system decays even more rapidly. Notice how similar figures 3.3a and 3.3b are, except in the $T_0$ and $T_1$ evolution.

This general pattern is confirmed in figure 3.4. This figure shows how the symmetric
initial structure $T(0, y) = \psi_4(y)$ and antisymmetric initial structure $T(0, y) = \psi_5(y)$ evolve spatially in the meridional direction from 0 to 300 days. Also shown in this figure is how the symmetric pattern would evolve if we artificially suppress the atmospheric Kelvin wave in the SST calculation. Notice how all pattern amplitudes look similar at the beginning. They all evolve equatorward, where the major changes are more apparent. The antisymmetric structure decreases in magnitude as it propagates toward the equator from day 0 to about day 150 (notice the decrease in the shading) until the SSTs projects onto $\psi_1(y)$ close to the equator where growth increases again (notice the shading starting at day 150). In contrast, Fig. 3.4b shows the amplitude of the symmetric pattern simply decaying with time. We compare this pattern with Fig. 3.4c which shows how the symmetric structure evolves without the atmospheric Kelvin wave. The damping effect of the Kelvin wave is apparent from about day 100: while the symmetric structure equatorial amplitude continues to decay in Fig. 3.4b, the pattern grows at the equator when no Kelvin wave is present (Fig. 3.4c). This is a contrast with the similarity between both patterns at the beginning, when the signal had not yet reached the equator.

Figure 3.5 shows the evolution of the SST modal amplitude $T_m^2$ for the same initial conditions considered above, including the evolution of the symmetric initial condition when we suppress the atmospheric Kelvin wave. Not only is the symmetric initial condition able to grow transiently in this case, its growth is actually bigger than the antisymmetric case. This is consistent with the values of the growth function $f(m = 0, \sigma_0, 0)$ when we zero out the Kelvin wave (red dots in Fig. 3.2b). This shows how powerful and determinant the atmospheric Kelvin wave is in affecting thermodynamically coupled mode dynamics. Without its influence thermodynamically coupled symmetric modes would actually grow more than equatorially antisymmetric ones.
3.3.3.1 Total SST growth

From the SST equation (equation 3.1) we calculate an equation for the growth of total SST anomalies. In this $\nu = 0$ limit (see appendix E (E.6) for the general growth equation), and considering equation 3.12, this is equal to

$$\frac{\partial < T^2 >}{\partial t} = L_x \sum_{m=0}^{\infty} \frac{\partial |T_m|^2}{\partial t} = L_x \varepsilon_T \sum_{m=0}^{\infty} f(m, \sigma, 0)|T_m|^2,$$

(3.15)

where the symbol $<>$ means integration over the basin considered, and $L_x$ is its zonal extension. The exchange terms in equation 3.12 ($\propto h$) get canceled out when we sum all the $\frac{\partial T_m^2}{\partial t}$ terms. The total SST growth evolution is described by this equation, together with equation 3.12 that determines the growth of an individual mode. Equations (3.15 and 3.12) indicate that in the $\nu = 0$ limit the role of the exchange function $h(m, 0)$ is to propagate the signal towards lower modes (curbing growth in the process, as explained in next section), and the growth function $f(m, \sigma, 0)$ determines whether that signal transiently grows or decays.

Figure 3.6 shows the total growth $< T^2 >$ for antisymmetric initial conditions $T(0) = \psi_1(y), \psi_3(y), \psi_5(y), \psi_7(y), \psi_9(y)$ (fig 3.6a), symmetric initial conditions $T(0) = \psi_0(y), \psi_2(y), \psi_4(y), \psi_6(y), \psi_8(y)$ (fig 3.6b), and the same symmetric initial conditions with the atmospheric Kelvin wave suppressed (fig 3.6c). We notice that the $T_1$ antisymmetric initial condition grows for some time (about 100$d$) before it is damped by $T_3$ (see 3.12). All the other antisymmetric initial conditions decay until the signal propagates close enough to the equator that it projects significantly onto $\psi_1(y)$, at which point the system experiences some growth, leading to secondary peaks that are less energetic and peak later in time (e.g. around 190$d$ for $T_3$ and 250$d$ for $T_5$). The symmetric initial conditions all decay, with $T_0$ decaying the fastest (due to the atmospheric Kelvin wave damping effect) and $T_2$ the slowest. This is consistent with the values of $f(m, \sigma_0, 0)$ shown in figure 3.2b. When the Kelvin wave is
suppressed the system is able to grow more effectively for initial conditions starting near the equator (lower order modes). Without the Kelvin wave, the SSTs are able to grow due to the positive effect of the atmospheric Rossby waves. This is consistent with the value of the growth function $f(m, \sigma_0, 0)$ with the Kelvin wave suppressed (figure 3.2b red dots).

### 3.3.3.2 Normal vs non-normal growth

In the limit $\nu \to 0$ we may write equation 3.10 in matricial form

$$\frac{\partial T}{\partial t} = MT = -i\omega T,$$

(3.16)

where $T = (T_0, T_1, T_2, \ldots)\exp(-i\omega t)$, and $M$ entries are given by

$$M_{m,m-2} = -\frac{1}{2} \epsilon_T h(m - 1, 0)\sigma$$
$$M_{m,m} = \frac{1}{2} f(m, \sigma, 0)$$
$$M_{m,m+2} = \frac{1}{2} \epsilon_T h(m + 1, 0)\sigma.$$

(3.17)

For $\sigma = \sigma_0$ the spectrum of $M$ is such that the system is linearly stable (i.e. $Im(\omega) < 0$ for all $\omega$) and non-normal (i.e. $M^T M \neq MM^T$), so growth may be achieved transiently (Farrell and Ioannou 1996). This also implies that $\lim_{t \to \infty} T = 0$ as is the case in previous plots.

The $\nu \to 0$ limit provides a convenient way to show the differences between normal and non-normal growth. The system is non-normal due to the exchange function $h$ being non-zero and of opposite sign for $M_{m+2,m}$ and $M_{m,m+2}$ (3.17). Artificially suppressing non-normality by zeroing out the exchange functions, the equation that determines the evolution
of a particular mode amplitude is now equal to (compare to equation 3.10):

\[ \frac{\partial T_m}{\partial t} = \frac{1}{2} \epsilon_T f(m, \sigma, 0) T_m. \]  

(3.18)

The equation for total SST growth (equation 3.15) remains unchanged, and actually contains the same information as (3.18). Note, however, that in the normal system (no exchange terms) each mode can be integrated individually using (3.18) and the result summed to obtain total growth (the integration of equation 3.15). The solution to equation 3.18 is

\[ T_m(t) = T_m(0) \exp \left( \frac{1}{2} \epsilon_T f(m, \sigma, 0) t \right). \]  

(3.19)

We observe that without the exchange terms the system will just decay or grow indefinitely depending on the sign of \( f \). That is, for \( \sigma = \sigma_0 \) the system will grow indefinitely if the initial SSTs project onto \( \psi_1(y) \) otherwise it will decay. In reality, the WES feedback coupling is “non-normal” so the growth is curbed by the exchange terms. In that case the SSTs will grow for some time if they project onto \( \psi_1 \), but the growth will be limited by the effect of the exchange terms.

Figure 3.7 shows the total growth for both normal (3.18) and non-normal (3.10) cases for the antisymmetric initial condition \( T(0) = \psi_1(y) \) and symmetric initial condition \( T(0) = \psi_0(y) \). In the normal case the antisymmetric initial condition grows exponentially, and the symmetric initial condition decays exponentially. In the non-normal case the growth of the antisymmetric initial condition initially exceeds growth of the normal system due to growth of \( T_3 \) and higher modes, but is eventually curbed by the interaction with \( T_3 \) (the exchange term \( \propto h(2, 0) \sigma_0 T_1 T_3 \) in equation 3.10). Similar arguments hold for the symmetric case \( T_0 \), which is additionally damped by \( T_2 \) and the system decays even more rapidly than the normal case. Thus, non-normality contributes to both short-term growth in the system, as
well as eventual decay.

3.3.4 Finite large scale zonal variability

The $k = 0$ limit is useful in understanding the low frequency large scale thermodynamically coupled variability. This section will describe how long but finite large scale variability deviates from this idealized case. As was done for the $k = 0$ case (equation 3.16) we can study the stability of the system by constructing a dynamical matrix that contains the WES feedback coupling (3.1) for a general $k$ value and analyze its eigenvalues. The matrix is similar to (3.17), but we use the general functions shown in (3.9). That is

$$M_{m,m-2} = -\frac{1}{2}\epsilon_T h(m - 1, \nu)\sigma$$

$$M_{m,m} = \frac{1}{2}f(m, \sigma, \nu)$$

$$M_{m,m+2} = \frac{1}{2}\epsilon_T h(m + 1, \nu)\sigma.$$  

(3.20)

Note that here we retain the non-normality inherent to the system. Solving equation 3.16 for this dynamical matrix, we find that an initial condition $T(0) = (T_0(0), T_1(0), T_2(0), ...)$ evolves as

$$T(t) = \exp(\mathbf{M}t)T(0) = \mathbf{G}(t)T(0).$$  

(3.21)

This shows that if all $\mathbf{M}$ eigenvalues $Im(\omega)$ are negative (see equation 3.16) the system is linearly stable and asymptotically decays (although the initial SST anomalies may grow for a finite time due to non-normality as shown in the previous section for $k = 0$). If at least one $\omega$ is positive, then the system is linearly unstable, and will be dominated by exponentially growing modes (akin to the normal case shown in figure 3.7).

Figure 3.8 shows $Im(\omega)$ for the least stable eigenvector of the system as a function
of \( \nu = \frac{k}{\pi} \) for equatorially antisymmetric case, symmetric case, and symmetric with the atmospheric Kelvin wave suppressed case. For context a zonal wavenumber \( k = \frac{2\pi}{120\sigma} \) and \( \epsilon = (2d)^{-1} \) corresponds to \( \nu = 2.44 \). For the standard parameters shown in table 3.1 the system is linearly stable for large zonal scale variability. The system does become unstable for larger \( k \) (and stable again for even bigger \( k \)) but that is well outside the validity of the Gill-Matsuno model under the long wave approximation. The system is stable at all \( k \) when the long wave approximation is not used (not shown, see section 3.3.5 for a description of the model used for that effect).

We observe that the stability of the system is greatly reduced for equatorially symmetric variability when we suppress the atmospheric Kelvin wave for very long zonal wavelengths up to \( \nu \sim 3.9 \). After that point the absence of the Kelvin wave does not seem to affect the stability of the system: note how the dashed and dash-dot curves converge in figure 3.8. The atmospheric Kelvin wave gets progressively less important in the dynamics of the system as \( \nu \) increases. Figure 3.9 shows that the amplitudes of the real part of the growth function \( f(m = 0, \sigma_0, \nu) \) with and without the Kelvin wave get closer for increasing \( \nu \). Consequently the destructive effect on growth due to the Kelvin wave gets less important as we deviate more and more from the \( k = 0 \) idealized case. This is simply a result of phasing of the Kelvin wave response with respect of the initial forcing: as \( k \) gets larger or \( \epsilon \) smaller, the Kelvin wave response becomes out of phase with the forcing (see equation 3.6). Both with and without the atmospheric Kelvin wave \( f(m = 0, \sigma_0, \nu) \), values should asymptote to \(-2\) (see equation 3.9) as \( \nu \rightarrow \infty \).

Overall, the picture gets much more complicated for non-zero zonal wavenumber. Now the evolution of a particular mode \( T_m \) will depend on the real and imaginary part of the growth function \( f(m, \sigma, \nu) \) as well as in the phasing in the complex plane with modes \( T_{m-2} \) and \( T_{m+2} \). Nonetheless, the general lessons learned in the \( k = 0 \) case remain valid (albeit
much less pronounced) for large scale dynamics.

Figure 3.10 shows the maximum transient growth for modes that are equatorially antisymmetric, symmetric, and symmetric with atmospheric Kelvin wave suppressed as a function of $\nu$. Only the linearly stable regime is considered. The framework used to calculate the optimal initial conditions that maximize transient growth in each case is described in appendix F [see also Farrel and Ioannou (1996), Vimont (2010), Martinez-Villalobos and Vimont (2016)]. We observe that the maximum growth for antisymmetric and symmetric variability starts converging as $\nu$ increases. The damping provided by the atmospheric Kelvin wave is still important, but relatively less so as the zonal wavelength gets shorter. This effect is emphasized by comparing the maximum growth with and without the Kelvin wave influence in figure 3.10.

The reason for the decrease of damping provided by the Kelvin wave as $\nu$ increases, and consequently the increase in similarity between symmetric and antisymmetric structures growth due to the WES feedback, is explained by the relative phasing between the SST and atmospheric response for a finite $\nu$. Figure 3.11 illustrates how the first Rossby wave contributes to growth, and the Kelvin wave contributes to damping for a zonal wavelength of $120^\circ$ ($\nu = 2.44$). Notice in figure 3.10 that for this $\nu$ antisymmetric structures still grow more than symmetric ones, but the gap has decreased tremendously compared to the $k = 0$ case. Consequently, we expect a diminished (compared to the $k = 0$ case), but still sizable contribution to damping on average by the Kelvin wave.

Figure 3.11a shows the instantaneous wind anomaly response to a sea surface temperature anomaly pattern of the form $T(x, y) = \psi_0(y)exp(ikx)$ associated with the Kelvin wave. Focusing first on the positive SST anomaly located from $x = 0.25$ to $x = 0.75$ ($x$ in $\frac{2\pi}{k}$ units) in the zonal axis, we notice that over the western part of this pattern (from $x = 0.25$ to $x = 0.45$) the total wind relaxes (a positive wind anomaly implies a relaxation of the
easterly trades), while in the eastern part (from $x = 0.45$ to $x = 0.75$) the total wind magnitude increases. The relaxation of the wind in the western part implies a decrease in evaporation and a tendency for the SST anomaly to grow there, while in the eastern part the reinforcement of the trades implies a cooling of the warm anomaly. In other words, there is SST growth in the western part and damping in the eastern part. The Kelvin wave acts to damp the anomaly on average because for this $\nu$ value the phasing is such that the negative wind anomaly associated with the Kelvin wave overlaps most of the warm anomaly (the eastern part is bigger in extension than the western part). A similar reasoning explains that on average the first Rossby wave acts to grow the anomaly. In this case, the reduced group velocity of the atmospheric Rossby wave means that the wind relaxes over most of the warm SST anomaly (figure 3.11b). Adding the responses (figure 3.11c) shows that in this regime the WES feedback acts to grow the initial anomaly. Notice also that this mechanism implies a westward propagation of the structure.

In the $\nu = 0$ limit the Kelvin and Rossby wave responses are exactly zonally in-phase with the SST anomalies. In that limit, both negative wind anomalies associated to the Kelvin wave, and positive wind anomalies associated to the first Rossby wave perfectly overlap a SST warm anomaly in the zonal direction, and SST anomalies grow or decay in place with no zonal propagation. For a finite zonal wavelength the phasing of the Kelvin wave changes more rapidly as $\nu$ increases than the phasing induced by the atmospheric Rossby wave, due to the differing group velocities. As a consequence, the Kelvin damping gets reduced more rapidly than the Rossby growth, so on average the symmetric SST mode grows more for shorter, but still long, zonal scales. In summary, in the zonal long wave range, as $\nu$ increases from 0 the equatorially symmetric mode propagates more efficiently westward achieving more growth in the process compared to smaller $\nu$.

The damping produced by the Kelvin wave is certainly smaller for finite $\nu = \frac{k}{\epsilon}$, but it
is still sizable. Figure 3.12b shows the evolution of the symmetric structure shown in figure 3.1a with the atmospheric Kelvin wave suppressed. The growth in this case is vastly larger (figure 3.12c, also compare Fig. 3.12b to Fig. 3.1c) showing the importance of this mechanism in damping symmetric structures. This explains why equatorially antisymmetric large-scale patterns are preferentially excited by the WES feedback: for symmetric structures, the Kelvin wave plays a critical role in damping variability, while there is no analogous mechanism for anti-symmetric structures.

3.3.5 The use of the long wave approximation

In any analytical approach some allowance is needed in order to find the adequate balance between analytical understanding and realism. In this study we take an analytical look on the equations that govern thermodynamically coupled variability in hopes of gaining insight on the differences and similarities between equatorially symmetric and antisymmetric modes. To simplify the mathematics and interpretation of the variability we use the long wave approximation. This approximation should give us a qualitative understanding of the variability in the vicinity of the $k = 0$ region. Here, we test the validity of the major conclusions away from the $k = 0$ limit by relaxing the long wave approximation.

Vimont (2010) and Martinez-Villalobos and Vimont (2016b) numerically analyze a Gill-Matsuno atmospheric model coupled to a slab ocean without using the long wave approximation under a variety of situations. We will refer to such a model as the “full model” as we will use it to contrast the results derived herein using the long wave approximation. In the full model we retain the atmospheric tendencies and add back atmospheric damping in the $v$ equation. [The difference between retaining or dropping the atmospheric tendencies is negligible for the $k$ range we are interested in, but solutions are much more easily computed using equation 5.3. See Vimont (2010) or Martinez-Villalobos and Vimont (2016b) for more
The resulting equations are projected onto the first 10 parabolic cylinder functions for consistency with the present analysis, and equatorially symmetric and antisymmetric terms are collected. The resulting equations may be written in the form of equation 3.16 with solution shown in (5.3), but in this case the state vector also contains the atmospheric variables.

Figure 3.13 shows the maximum SST growth of the first symmetric and antisymmetric optimal as a function of $\nu$ under the full model. We compare this figure with figure 3.10a that shows the same curves (symmetric and antisymmetric) but under the long wave approximation. At first look the symmetric and antisymmetric maximum growth in both models don’t look similar, but a closer look reveals more similarities than differences. The symmetric optimal does not experience growth for small $k$, and starts to grow roughly at the same value of $\nu \sim 1$. The antisymmetric optimal maximum growth is always bigger than the maximum symmetric growth (for this parameter regime), and both maximum symmetric and antisymmetric growth approach each other as $k$ increases, likely because the Kelvin wave also kills growth in the full model and its effect is reduced for shorter zonal scales.

The symmetric variability is very well approximated at small $k$ while there is an error in the antisymmetric variability even at small $k$ (the $\epsilon v$ term dominates the meridional momentum equation very close to the equator for antisymmetric variability at any $k$, while for the symmetric mode $\epsilon v$ is strictly zero at the equator and small in the vicinity of the equator). In many respects the long wave approximated model acts like a less damped version of the full model, which explains the smaller growth shown in general in Fig. 3.13 compared to figure 3.10a. That parallel is not exact for the antisymmetric variability. In the full model the maximum growth occurs at $k = 0$ in agreement with Xie et al. 1999 while in the long wave approximation the maximum growth occurs for $\nu > 0$. In spite of these differences the spatial structures of the symmetric and antisymmetric optimals (and
also the time evolved structures) for the full and long wave approximated model look almost identical, showing that the long wave approximation is a good representation of the WES feedback process, despite differences in the maximum growth attained (less damped than the full model).

### 3.4 Conclusions and final comments

#### 3.4.1 Conclusions

The growth and equatorial symmetry properties of thermodynamically coupled ocean-atmosphere variability was studied using the Gill-Matsuno atmospheric model coupled to a thermodynamic slab ocean. Assuming an atmosphere completely determined by the underlying SSTs the ocean and atmosphere variables were projected onto parabolic cylinder functions $\psi_m(y)$, effectively decomposing the variability onto different $T_m$ modes. Under the assumption of a geographically homogeneous coupling (i.e. $K_q$ and $\alpha$ constant) two independent sets of solutions emerge: equatorially symmetric ($m$ even), and equatorially antisymmetric (meridional mode-like, $m$ odd). These two sets of solution behave similarly away from the equator, for $T_m$ modes $m \geq 2$, and differ significantly in the equatorial zone, for $T_m$ modes $m = 0, 1$.

The main difference in the equatorial zone is traced to the additional SST damping provided by the atmospheric Kelvin wave for equatorially symmetric variability for large zonal scales. On the other hand antisymmetric variability close to the equator is unaffected by a damping term analogous to the Kelvin one for symmetric variability, leading to more growth on average.

It is found that both equatorially symmetric and antisymmetric variability propagates in a similar manner towards the equator. In this framework this is realized by a decrease in amplitude of higher $T_m$ modes and an increase in amplitude of lower $T_m$ modes. When
the variability reaches the equatorial zone it excites the $m = 0$ mode in the symmetric case, and the $m = 1$ mode in the antisymmetric case, leading to two very different outcomes. In the symmetric case the SST signal is subjected to additional damping by the atmospheric Kelvin wave, while in the antisymmetric case the SST signal grows through the positive WES feedback produced by the first antisymmetric atmospheric Rossby wave. For large zonal scales the outcome of this process is growth of the antisymmetric mode, and decay of the symmetric mode. In other words any SST distribution will grow effectively through this mechanism if its meridional structure has a large projection onto the $\psi_1(y)$ function (equatorially antisymmetric SST structure peaking around 8° of latitude for the parameters in this study, also see figure C.1). On the other hand an equatorially symmetric SST distribution confined to the equator [large projection onto the $\psi_0(y)$ function] will be short lived in absence of other feedbacks. For shorter zonal wavenumber, but still relatively large zonal scale, damping of symmetric structures by the Kelvin wave is reduced and the growth of both symmetric and antisymmetric variability is similar.

Under the range of parameters considered in this study the growth process of thermodynamically coupled variability is fundamentally non-normal. This non-normality is manifest in interactions that initially grow a SST anomaly while at the same time seeding its eventual decay. Initial growth occurs as a given SST structure (without loss of generality, we consider an initial positive anomaly) produces a positive tendency for lower-order modes, and negative tendency for higher order modes. These tendencies lead to short-term growth. The positive tendency of the lower-order mode excites the next-lowest-order mode, and so forth until the lowest order anti-symmetric ($m = 1$) or symmetric ($m = 0$) structure is excited, at which that mode either grows on its own (for the $m = 1$ anti-symmetric structure) or experiences enhanced decay due to the Kelvin wave (for the $m = 0$ symmetric structure, as described above). At the same time (again assuming initial positive polarity for a given
mode), as the amplitude of the higher order mode grows more large and negative, the negative amplitude of the higher order mode in turn contributes to decay of the original SST mode. The non-normal decay process eventually counteracts growth of the total anomaly, and is critical for maintaining stability of the system (under the parameter regime considered herein). In principle this growth process does not require equatorial antisymmetry and in fact it would prefer the symmetric structure should the Kelvin wave not exist.

Using this framework we show that both equatorially symmetric and antisymmetric modes propagate westward as a consequence of the group velocity of atmospheric Rossby waves. In the antisymmetric case the propagation is due to the zonal phasing between the antisymmetric Rossby wave wind response and SST anomalies as found by Xie et al. 1999. In the symmetric case, the interpretation is the same for $m > 2$ symmetric Rossby waves, except in the equatorial zone. In the equatorial region the propagation direction will depend on the relative phasing between the first atmospheric Rossby and Kelvin waves. It turns out that for large zonal scales, this phasing favors the growth by the first Rossby wave implying a westward propagation of the SST structure.

3.4.2 Discussion

The implications of this study, although assuming a homogeneous coupling, can be extended into a system that includes geographical variations in the mean state, as manifest by a geographic dependence of $K_q$ (Martinez-Villalobos and Vimont 2016b). In that case, atmospheric heating depends on the product $K_q(y)T$, and hence the mean state acts as a “mode selector” for enhancing coupling of particular modes. As an example consider the air-sea coupling distribution provided by an equatorially symmetric but meridionally thin Inter Tropical Convergence Zone (ITCZ). This coupling structure will suppress off-equatorial modes and enhance just the atmospheric Kelvin and lowest-order Rossby waves. In the limit
of a vanishingly thin ITCZ (in which $K_q(y)$ approaches a delta-function), coupling for antisymmetric modes are eliminated (via the product of $K_q(y)$ and antisymmetric $T_m$) and we would expect just a thermodynamically coupled symmetric mode to be able to grow, as shown in Martinez-Villalobos and Vimont (2016). Following the same idea, a broader symmetric ITCZ would produce a situation more akin to the one presented in this paper: more growth for antisymmetric modes, because the symmetric modes are additionally damped by the Kelvin wave. This interpretation could be extended to include variations in the assumed atmospheric radius of deformation as well, e.g. the case of a Battisti et al. (1999) or Lindzen and Nigam (1987) style model of the atmospheric boundary layer. Similar arguments could also be applied to understanding zonal evolution through a zonally and meridionally varying mean state.

This framework also helps understand the consequences of equatorial asymmetry in the coupling, for example through meridional asymmetries in the ITCZ position. As discussed above, a narrowly confined equatorially symmetric ITCZ (e.g. tropical Atlantic during boreal spring) suppresses coupling for off-equatorial and anti-symmetric modes and enhances the atmospheric Kelvin and first symmetric Rossby responses. In contrast, for a narrow ITCZ placed off the equator (e.g. tropical Atlantic and eastern Pacific during boreal fall) the coupling is asymmetric. In that case, preference for equatorially symmetric or antisymmetric modes will depend on the relative projections of the $\psi_1(y)$ and $\psi_0(y)$ structures onto $K_q(y)$. For a narrow ITCZ structure centered near a maxima in $\psi_1(y)$, antisymmetric modes will dominate the solution (even with a non-zero projection of $K_q(y)$ onto $\psi_0(y)$) due to the damping effect of the Kelvin wave for the symmetric component of the solution. In that case however, the solution will include a mix of both symmetric and antisymmetric components. This mechanism explains the findings of Martinez-Villalobos and Vimont (2016b) in that regard (compare figures 4 and 5 in that paper). This interpretation may also be useful for
interpreting similar variability in models with less restrictive coupling parameterizations, e.g. Lindzen and Nigam (1987), Battisti et al. (1999).

This work shows that both anti-symmetric and symmetric structures emerge from the physical processes that generate tropical “meridional modes”; i.e. both anti-symmetric and symmetric structures grow and propagate due to the same physical processes. This is somewhat in contrast to early discussions of meridional mode behavior, where it was thought that equatorial anti-symmetry would enhance growth through providing an interhemispheric temperature gradient that would enhance the surface wind response. Given the results herein, it appears that an interhemispheric gradient is not necessary, and in fact equatorially symmetric structures would preferentially grow in the absence of the Kelvin wave. Instead, growth is more closely related to the zonal and meridional phasing between the atmospheric Rossby wave response to imposed heating and the SST that is presumably responsible for that heating in the first place (Vimont 2010). Based on these arguments, one might ask whether the term “meridional mode” is still appropriate. We show herein that both anti-symmetric and symmetric structures exist as a class of structures that evolve in a similar manner towards lower meridional wavenumbers; as such, we argue that the phrase “meridional mode” is appropriate for both anti-symmetric and symmetric structures.

The aim of this study is to explain the main similarities and differences between symmetric and antisymmetric thermodynamically coupled variability using a common framework. As such, besides the usual linearity assumption, there are many approximations being made and a lot of room to improve the model. For example, the parameters in the model, especially the coupling, do not vary geographically. There are different regions in the tropical oceans where we would not expect the same WES feedback coupling. Another important approximation is the assumption that latent heat is unaffected by meridional wind variations. These variations will affect antisymmetric and symmetric modes differently, and we hope to
consider them in a subsequent study. Also, the use of the long wave approximation produces less damped modes than using a model with meridional wind damping, although that does not affect the spatial structure of the modes. Finally variations of other fluxes, specially short wave flux, should be considered in the future. Despite these caveats, this study does provide physical insight into the workings of thermodynamically coupled variability.

In nature, equatorially antisymmetric modes coupled through the WES feedback are an established and relevant part of our climate system, and are the subject of a large literature collection studying meridional mode variability in both the tropical Atlantic and Pacific. On the other hand, thermodynamic equatorially symmetric variability has been less studied, likely because the Bjerknes feedback masks these kinds of modes. Nonetheless the interest in this kind of variability has increased following the findings of Clement et al. (2011). In addition the Pacific Meridional Mode has an important degree of equatorial symmetry (Chiang and Vimont 2004), and as such the results derived herein may be relevant in nature. We hope that this study provides a common ground to understand these modes in a unified framework.

Acknowledgments.

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Table 3.1: Model parameters

<table>
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<tr>
<th>Parameters</th>
<th>Value</th>
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<tbody>
<tr>
<td>$\rho_o$</td>
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</tr>
<tr>
<td>$c_o$</td>
<td>$4.2 \times 10^3\ J\ kg^{-1}\ K^{-1}$</td>
</tr>
<tr>
<td>$H_o$</td>
<td>50\ m</td>
</tr>
<tr>
<td>$\hat{\alpha}$</td>
<td>$10\ J/m^3$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\frac{1}{\rho_o c_o H_o}$</td>
</tr>
<tr>
<td>$K_q$</td>
<td>$1.7\ m^2/K$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>$2\ d^{-1}$</td>
</tr>
<tr>
<td>$\epsilon_T$</td>
<td>$120\ d^{-1}$</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>4.83 (non-dim)</td>
</tr>
</tbody>
</table>
Figure 3.1: (a) Optimal symmetric and (b) anti-symmetric initial conditions that maximize growth at 180 days. The 180 day evolution of these initial conditions are shown in (c) for the symmetric optimal and (d) for the anti-symmetric optimal. (e) Growth of both optimal initial conditions (note that only SST contributes to state growth). For these plots the wavelength $L_x = 120^\circ$, and $L_y$ is the equatorial radius of deformation ($\sim 10^\circ$ for the parameters in table 3.1). Units are arbitrary, but consistent for all panels. SST is represented by shading, and low level winds by the arrows.
Figure 3.2: (a) The exchange function $h(m, 0)$ as a function of SST mode $m$. This function monotonically increases towards its asymptotic value of $\frac{1}{2}$. (b) The growth function $f(m, \sigma, 0)$ as a function of $m$ for different values of the stability parameter $\sigma$. Values in red show $f(m = 0, \sigma, 0)$ when the atmospheric Kelvin wave is suppressed.
Figure 3.3:  (a) Evolution of individual $T_m$ coefficients for the symmetric initial condition $T(0) = \psi_4(y)$.  (b) Evolution of individual $T_m$ coefficients for the antisymmetric initial condition $T(0) = \psi_5(y)$. Both panels are shown for stability parameter $\sigma_0$ and for $k = 0$. 
Figure 3.4: Latitude-time evolution of SST from 0 to 300 days. (a) Evolution of the anti-symmetric initial condition \( T(0) = \psi_5(y) \). (b) Evolution of the symmetric initial condition \( T(0) = \psi_4(y) \). (c) Same as (b) except that the atmospheric Kelvin wave is suppressed. Shading represents SST. In this \( k = 0 \) case the evolution does not depend on longitude.
Figure 3.5: $|T_m|^2$ evolution for: (a) the antisymmetric initial condition $T(0) = \psi_5(y)$, (b) the symmetric initial condition $T(0) = \psi_4(y)$, and (c) $T(0) = \psi_4(y)$ [as in (b)] except that the atmospheric Kelvin wave is suppressed. All panels are shown for stability parameter $\sigma_0$ and for $k = 0$. 
Figure 3.6: Total SST growth for (a) antisymmetric initial conditions, (b) symmetric initial conditions, and (c) symmetric initial conditions [as in (b)] except that the atmospheric Kelvin wave is suppressed. For panel (a) the lines show the total growth for initial conditions starting at $T_1$ (solid), $T_3$ (dashed), $T_5$ (dotted), $T_7$ (dash-dot), and $T_9$ (+ sign). For panels (b) and (c) the lines show the total growth for initial conditions starting at $T_0$ (solid), $T_2$ (dashed), $T_4$ (dotted), $T_6$ (dash-dot), and $T_8$ (+ sign). All panels are shown for stability parameter $\sigma_0$ and for $k = 0$. 
Figure 3.7: Total SST growth for the non-normal system (solid line; equation 3.10), and for the “normalized” system (dashed line; equation 3.18; see text for details). (a) Growth from antisymmetric $T(0) = \psi_1(y)$ initial conditions. (b) Growth from symmetric $T(0) = \psi_0(y)$ initial conditions. Both cases are shown for stability parameter $\sigma_0$ and for $k = 0$. 
Figure 3.8: Growth rate ($\times 10^{-3} \, s^{-1}$) of the least stable eigenmode of $M$ (equation 3.20) as a function of $\nu$. $Im(\omega) > 0$ implies a linearly unstable mode. The parameter $\nu$ is defined as $\nu = \frac{k}{\epsilon}$. For the parameters shown in table 3.1, a wavelength $L_x$ of 120° corresponds to $\nu = 2.44$. 
Figure 3.9: Real part of $f(m = 0, \sigma_0, \nu)$ as a function of $\nu = \frac{k}{\ell}$. The solid line contains both the effect of the atmospheric Kelvin and first Rossby waves on $f(m = 0, \sigma_0, \nu)$, while the dashed line contains just the first Rossby wave. Notice that the curves converge as $\nu$ increases, indicating a decreasing influence of the atmospheric Kelvin wave with increasing $\nu$. 
Figure 3.10: (a) Maximum growth $\mu$ as a function of $\nu$. (b) Time when maximum growth is achieved as a function of $\nu$. The plots cover the linearly stable range under the long wave approximation.
Figure 3.11: Instantaneous atmospheric wind (vectors) and low level geopotential (contours) response to $SST = \psi_0(y) \exp(ikx)$. Shown are the: (a) Kelvin wave, (b) first Rossby wave, and (c) total instantaneous atmospheric response. Plots correspond to a zonal wavenumber $L_x = 120^\circ$ corresponding to $\nu = 2.44$, and $L_y \sim 10^\circ$ (from parameters in Table 3.1). SST is shaded, and the solid (dashed) contours correspond to positive (negative) low level geopotential anomalies. Units are arbitrary, but consistent between panels.
Figure 3.12: (a) Symmetric “optimal” initial condition from Fig. 3.1(a). (b) 180 day evolution of the initial condition from (a) for the case when the atmospheric Kelvin wave is suppressed [compare to figure 3.1(c)]. (c) Growth of the 180 day symmetric optimal with and without the Kelvin wave. For these panels $L_x = 120^\circ$ $L_y$ (the equatorial radius of deformation) $\sim 10^\circ$. Shown is SST (shaded) and low level winds (vectors). Units are arbitrary, but are the same in panels (a), (b), and Fig. 3.1(a-c).
Figure 3.13: Maximum growth of equatorially symmetric and antisymmetric variability under full model (no long-wave approximation). Compare to figure 3.10.
A method to estimate the parameters of a Linear Inverse Model driven by Correlated Additive-Multiplicative noise

In collaboration with Matthew Newman, Cécile Penland, and Daniel J. Vimont

Abstract

In this short note we introduce a method to fit a multidimensional non-Gaussian dataset to a Correlated Additive-Multiplicative (CAM) noise driven Linear Inverse Model (CAM-LIM). This CAM-LIM formulation conserves the virtues of the regular forced by state independent white noise Linear Inverse Model (LIM) such as, i.e the expected conditional evolution remains dictated by linear dynamics, and the same data covariance structure is described, while at the same time generating skewness and excess kurtosis. To simplify the calculations we introduce two important approximations: i) The CAM noise is ”diagonal”, and ii) The multiplicative noise depends only on the variable being integrated. This formulation updates the 1-D constraint between skewness $S$ and excess kurtosis $K$ ($K \geq \frac{3}{2}S^2$) to include the effects of the deterministic coupling and off-diagonal pure additive noise terms. The retrieval algorithm is tested in a 2 by 2 toy model with the following conclusions: i) Although the multilinear constraint is met in the “long term”, it may be violated over “short” intervals,
as short as 500 years in this particular model. ii) There is great variability on the parameters retrieved, even when comparing different 10000 year intervals. Not surprisingly this variability increases the shorter the segment considered. The spread is centered around the true parameter values. iii) There is also great variability in the skewness and excess kurtosis generated. Practical issues related to the implementation of this method on true climate variables are discussed.
4.1 Introduction

Multi-linear theory has been used to great success in practically all realms of climatic science. One of the most useful and widely applied linear method is the Linear Inverse Model (LIM) (Penland and Sardeshmukh 1995). In this model a linearly stable system, describing the evolution of a “slow” variable (like the ocean sea surface temperature), is driven by additive white noise, representing the effect of unresolved “fast” variability on the slow variable (Papanicolaou and Kohler 1974; Penland 1996). This kind of model can be used to forecast to great success (Newman 2013), and performs well when the underlying slow deterministic dynamics is linear or weakly non-linear.

Despite the qualitative (and often quantitative) success of linear methods, these kind of models are unable in general to reproduce the higher statistical moments of the system, such as skewness and kurtosis. The skewness quantifies the asymmetry of the probability density function (pdf) around its mean, and kurtosis quantifies how heavy or light the tails of a pdf are. Skewness and excess kurtosis (excess compared to a Gaussian distribution) may be generated through different dynamical mechanisms. One of these mechanisms is the forcing of a deterministic linear system by Correlated Additive-Multiplicative (CAM) Gaussian white noise (Sardeshmukh and Sura 2009; Penland and Sardeshmukh 2012). In this formulation the noise depends linearly on the state of the system, which modifies the kurtosis of the pdf compared to the Gaussian case, and this dependence is asymmetric with respect to the mean, which creates skewness. CAM noise has been shown to occur naturally in quadratically non-linear systems with a clear separation between slow and fast timescales (Sardeshmukh and Sura 2009) making its study highly relevant given the ubiquitousness of quadratic fluxes in the climate system. The one dimensional version of this model (Sardeshmukh and Sura 2009) predicts that the “true” skewness $S$ and excess kurtosis $K$ of the system are related
such that

\[ K \geq \frac{3}{2} S^2. \]  

(4.1)

Several variables have been found to follow such a parabolic \( K \geq \frac{3}{2} S^2 - \delta \) relationship (Sardeshmukh and Sura 2009; Sardeshmukh and Penland 2015; Sardeshmukh et al. 2015), where \( \delta > 0 \) is a small offset. For example figure 4.1 shows the skewness (fig. 4.1a) and excess kurtosis (fig. 4.1b) of monthly sea surface height (SSH) in the tropical Pacific over the 1958-2011 period. Also shown is the 1-D constraint \( K - \frac{3}{2} S^2 \) (fig. 4.1c). It is clear that high excess kurtosis regions correspond very closely to highly skewed regions.

The purpose of this short note is to introduce some consistency relations between the CAM-LIM parameters and the statistics generated by it, and use said relations to empirically derive the parameters from data. In order to do this we use the Fokker-Planck equation (Fokker 1914; Kolmogoroff 1931)

\[
\frac{\partial p(\vec{x}, t)}{\partial t} = \sum_i \frac{\partial (A_i(\vec{x}, t)p(\vec{x}, t))}{\partial x_i} + \frac{1}{2} \sum_{i,j,m} \frac{\partial B_{im}(\vec{x}, t)}{\partial x_j} B_{jm}(\vec{x}, t)p(\vec{x}, t) \\
+ \frac{1}{2} \sum_{i,j,m} \frac{\partial^2}{\partial x_i x_j} (B_{im}(\vec{x}, t)B_{jm}(\vec{x}, t)p(\vec{x}, t)),
\]  

(4.2)

which is the equation satisfied by the pdf of a deterministic system driven by white noise:

\[
\frac{dx_i}{dt} = A_i(\vec{x}, t) + \sum_m B_{im}(\vec{x}, t)\zeta_m.
\]  

(4.3)

In this equation \( A_i \) encodes the deterministic dynamics, \( B \) the noise amplitudes, and \( \vec{\zeta} \) is a vector of white noise processes. For future reference we will clarify the terminology used in (4.2). The first term in that equation corresponds to the “deterministic drift”, the second term is known as the “noise induced drift” and is zero if the noise is independent of the state of the system, and the last term is usually called the “diffusion”. We will illustrate
the derivation of the CAM-LIM parameters by using a 2 by 2 toy model and will discuss the practical issues associated to it. The remaining of this note is organized as follows. Section 4.2 presents a brief overview of the LIM framework. Section 4.3 presents the CAM-LIM, several important approximations, and the derivation of the parameters of the model as a function of its statistical structure. Section 4.4 exemplifies this in a $2 \times 2$ toy model, and compares it to the regular LIM case. Finally, section 4.5 presents the conclusions.

### 4.2 Brief Review of Linear Inverse Modeling

The most widely used stochastically generated system (SGS) is the one known as Linear Inverse Model (Penland and Sardeshmukh 1995). In this framework the $n$ components of the state of the system $\mathbf{x}$ evolve according to the following linear equation (also written in component notation for future use):

$$
\frac{d\mathbf{x}}{dt} = \mathbf{Mx} + \mathbf{B}\eta
$$

$$
\frac{dx_i}{dt} = \sum_j M_{ij}x_j + \sum_l B_{il}\eta_l.
$$

In this equation $\mathbf{M}$ is a constant $n \times n$ matrix, $\mathbf{B}$ is a $n \times m$ matrix of noise amplitudes, and $\mathbf{\eta}$ is a $m$ component vector of Gaussian white noise processes. The matrix $\mathbf{M}$ denotes the predictable linearized dynamics and all its eigenvalues are negative implying that the system needs the stochastic forcing to generate variance. There is one key difference in how this multilinear system behaves compared to its 1-D version. In absence of stochastic forcing the 1-D system decays exponentially, while in the multilinear case short-term growth is possible if the dynamics of the system is non-normal (Penland and Sardeshmukh 1995; Farrell and Ioannou 1996; Martinez-Villalobos and Vimont 2016b). This makes possible the use of this framework as a forecasting tool (Penland and Sardeshmukh 1995; Penland 1996; Newman
Using the Fokker-Planck equation applied to the conditional pdf, the expected evolution of the state of the system $\vec{x}(0)$ is given by

$$\vec{x}(t) = e^{Mt}\vec{x}(0).$$

(4.6)

There are conservation laws in the dynamics of SGSs that arise from the Fokker-Planck Equation. In particular the Fluctuation-Dissipation Relation (Penland and Matrosova 1994) relates the data covariance structure $C_0 = <\vec{x}\vec{x}^T>$, where $<>$ denotes a long term average, to the noise processes covariance structure $Q = BB^T$ as (where we also write this relation in component notation for future reference):

$$MC_0 + C_0M^T + Q = \frac{dC_0}{dt}$$

(4.7)

$$\sum_l (M_{nl} < x_l x_k > + < x_n x_l > M_{kl}) + \sum_mB_{nm}B_{km} = \frac{d < x_n x_k >}{dt}.$$  

(4.8)

The LIM framework is and has been used extensively to study the state of the tropical Pacific (Penland and Sardeshmukh 1995; Penland 1996; Newman et al. 2011), tropical Atlantic (Penland and Matrosova 1998; Vimont 2012), etc. In the tropical Pacific the forecast of SST anomalies through this method (by using equation 4.6) is competitive compared to forecasts provided by General Circulation Models (Newman and Sardeshmukh 2016). The LIM framework is good at describing the covariance structure, but in general is not able to describe the skewness and excess kurtosis of real systems.
4.3 Correlated Additive-Multiplicative noise Linear Inverse Model (CAM-LIM)

In order to retain the advantages of the LIM approach, and also get a good representation of the higher statistical moments of the system, we present a multilinear generalization of the CAM-LIM 1-D model introduced by Sardeshmukh and Sura 2009. This model accounts for slow-fast interactions, i.e. the effect of unresolved variability that is modulated by the state of the system, on the system trajectory (Sardeshmukh and Penland 2015). In component form the state of the system \( \vec{x} \) will evolve through the following equation

\[
\frac{dx_i}{dt} = \sum_j A_{ij}x_j + \sum_k B_{ik}\eta_k + \sum_l (G_{il} + \sum_m E_{ilm}x_m)\xi_l - \frac{1}{2} \sum_n \sum_h E_{inlh}G_{nh}.
\]

Here the matrix \( A \) encodes the linearized deterministic dynamics, the tensor \( E \) the amplitude of the multiplicative noise (also known as state dependent noise) components, the matrix \( G \) the amplitude of the additive noise that is correlated to the multiplicative noise, \( B \) the additive noise uncorrelated to it, and \( \vec{\eta} \) and \( \vec{\xi} \) denote two independent vectors of noise processes. For the time being we will consider all matrices to be time independent. The system as it is will create a mean response that needs to be negated by the last term in (4.9) in order to describe departures respect to the mean. We will relate \( A \) and \( E \) in 4.9 to \( M \) in 4.5 shortly.

At this point we will introduce two important approximations that will make the system much more tractable and simpler, and allows a relatively simple determination of the noise parameters \( E, G, \) and \( B \) as a function of the system statistics. The first one is that we will consider just one CAM noise process affecting each component of the state system (i.e. the multiplicative noise is "diagonal"). Without loss of generality we will label that process by
the label of the state component it affects, i.e. \( E_{imi} = E_{imi} \). The second approximation is that the state dependent noise part will only depend on the the same variable as the one being predicted, i.e. \( E_{imi} = E_{iii} \). (In a physical LIM, this corresponds to the “local” approximation). We hope to consider a more complete CAM noise formulation in the future. Multiplying the Fokker-Planck equation by the appropriate moment-generating function and integrating from \(-\infty\) to \(\infty\) we can calculate an equation for the first two moments of the system:

\[
\begin{align*}
\frac{d < x_k >}{dt} &= M_{kl} < x_l > \\
\frac{d < x_n x_k >}{dt} &= \sum_l (M_{kl} < x_l x_k > + < x_n x_l > M_{kl}) + \sum_m B_{nm} B_{km} + G_{nk}^2 \delta_{nk} + E_{nnn}^2 < x_k^2 > \delta_{nk}
\end{align*}
\]

(4.10)

where for analogy with the usual LIM notation (4.5), (4.8), and also for shortness we have defined

\[
M_{kl} = A_{kl} + \frac{1}{2} E_{kkk} \delta_{kl}.
\]

(4.11)

Using the Fokker-Planck equation applied to the conditional pdf we find that the most probable evolution of the system (under a root mean squared measure), given a current state \( \bar{x}(0) \) is also given by (4.6)

\[
\bar{x}(t) = e^{Mt} \bar{x}(0)
\]

(4.12)

which further justifies the use of the notation shown in (4.11). Also (4.11) shows that in general when calculating the matrix \( M \) from data, that determination not only includes the linearized deterministic drift, but also the noise-induced drift that masks itself as deterministic dynamics (Penland and Matrosova 1994). Finally, we point out that 4.12 includes the important conclusion that the predictable growth characteristics are the same as the standard LIM.
Equation 4.10 generalizes the fluctuation-dissipation relation to include the extra CAM noise terms. From the Fokker-Planck equation we also calculate an equation for the (unnormalized) skewness \( < x_k^3 > \) and kurtosis \( < x_k^4 > \) of the system:

\[
\frac{d < x_k^3 >}{dt} = 3 \sum_l M_{kl} < x_l x_k^2 > + 6E_{kkk}G_{kk} < x_k^2 > + 3E_{kkk}^2 < x_k^3 >
\]

\[
\frac{d < x_k^4 >}{dt} = 4 \sum_l M_{kl} < x_l x_k^3 > + 6(\sum_m B_{km}^2 + G_{kk}^2) < x_k^2 > + 12E_{kkk}G_{kk} < x_k^3 > + 6E_{kkk}^2 < x_k^4 >
\]

(4.13)

(In passing we can show that

\[
\frac{d < x_n^k >}{dt} = n \sum_l M_{kl} < x_l x_k^{n-1} > + \frac{1}{2}n(n - 1)(\sum_m B_{km}^2 + G_{kk}^2) < x_k^{n-2} >
\]

+ \( n(n-1)E_{kkk}G_{kk} < x_k^{n-1} > + \frac{1}{2}n(n - 1)E_{kkk}^2 < x_k^n > \). \quad (4.14)

Assuming stationarity we may find an expression for the parameters of the CAM-LIM system (4.9) in terms of its statistical moments as

\[
E_{kkk}^2 = - \sum_l M_{kl} \frac{2 < x_l x_k^3 > < x_k^2 > - 6 < x_l x_k^2 > < x_k^2 >^2 - 3 < x_l x_k > < x_k^3 >}{3 < x_k^4 > < x_k^2 > - < x_k^2 >^3 - < x_k^3 >^2}
\]

(4.15)

\[
G_{kk} = -\frac{1}{2}(\frac{E_{kkk}^2 < x_k^3 > + \sum_l M_{kl} < x_l x_k^2 >}{E_{kkk} < x_k^2 >})
\]

(4.16)

\[
Q_{kk} = -2 \sum_l M_{kl} < x_l x_k > - G_{kk}^2 - E_{kkk}^2 < x_k^2 >
\]

(4.17)

where we have defined \( Q = BB^T \). The non-diagonal elements of \( Q \) are calculated using (4.10).

It is immediately apparent that given \( E, G, \) and \( Q \) the CAM-noise formulation will only be able to access a subset of the moments space of the system. In particular the requirement
that (4.15) be positive imposes a hard constrain in the system. The denominator in (4.15) is always positive (Wilkins 1944), so this implies that

\[ \sum_i M_{kl}(2 < x_i x_k^3 > < x_k^2 > - 6 < x_i x_k > < x_k^2 >^2 - 3 < x_i x_k^2 > < x_k^3 >) < 0. \] (4.18)

This constraints reduces to (4.1) in the 1-D case, which is a good consistency check.

### 4.4 2 by 2 system example

In this section we look at how the retrieval algorithm (4.15), (4.16), (4.17) performs in practice, and the nuances associated to it. We consider a 2 by 2 version of (4.9) shown below

\[
\begin{align*}
\frac{dx_1}{dt} &= A_{11}x_1 + A_{12}x_2 + B_{11}\eta_1 + B_{12}\eta_2 + (G_{11} + E_{111}x_1)\xi_1 - \frac{1}{2}E_{111}G_{11} \\
\frac{dx_2}{dt} &= A_{21}x_1 + A_{22}x_2 + B_{21}\eta_1 + B_{22}\eta_2 + (G_{22} + E_{222}x_2)\xi_2 - \frac{1}{2}E_{222}G_{22}
\end{align*}
\] (4.19)

with the following parameters (there is no particular reason to choose these parameters):

\[
A = \begin{pmatrix}
-0.19 & -0.03 \\
0.10 & -0.09
\end{pmatrix} \quad B = \begin{pmatrix}
0.07 & -0.02 \\
-0.02 & 0.15
\end{pmatrix}
\] (4.20)

\[
G = \begin{pmatrix}
0.02 & 0 \\
0 & -0.06
\end{pmatrix} \quad E_{111} = 0.20 \quad E_{222} = 0.14.
\] (4.21)

We perform 10 runs of 100,000 years each using the Heun method (Rüemelin 1982) with a 3 hour integration time step, and we take and collect monthly averages (1 month = 30 days). In total the joint data consists of 12 million months. Removing the first 1000 or 10000 years of each run as spin up data has a negligible impact in the results, for which reason we have
kept the whole data set.

As a visual example figure 4.2 shows the pdf of the components of the state of the system $x_1$ (fig 4.2a), and $x_2$ (fig 4.2c) for the whole joint data set, and it compares it with the pdf of a similarly run dataset using the regular LIM (4.5), with $M_{it} = A_{it} + \frac{1}{2} E_{it}^2 \delta_{it}$, and constructed so both systems produce the same data covariance structure. In addition to that, table 4.1 shows both variables skewness and kurtosis under the regular LIM and CAM-LIM cases. The deviations from Gaussianity are evident, in specific the positive (negative) skewness of $x_1$ ($x_2$), and the heavier tails in the CAM-LIM case (figures 4.2b and 4.2d). This occurs despite both systems sharing the same conditional evolution (4.6), (4.12).

Table 4.1 also shows the theoretical skewness and kurtosis of a similar uncoupled 1-D CAM-noise (Sardeshmukh and Penland 2015) system (i.e. the cross terms $A_{12}, A_{21}, B_{12}$ and $B_{21}$ are 0 in (4.19)). As expected, the coupling modifies the statistical moments of the system, and that modification may be modest, as it is the case for $x_1$, or more pronounced, as in the $x_2$ case, depending on the parameters of the system. Repeating the integration of (4.19) but with only one variable affected by CAM noise ($E_{111} = G_{11} = 0$; for clarity we also take the off diagonal pure additive noise $B_{12}$, and $B_{21}$ terms as zero) reveals that the deterministic coupling is able to propagate the CAM noise influence from the second variable to the first (not shown).

4.4.1 Retrieval of the noise coefficients

In this subsection we analyze how well the retrieval algorithm (4.15), (4.16), (4.17) performs on the 2 by 2 system, with emphasis on the amount of data needed to get a good representation of its statistical structure, and consequently the ability to calculate the parameters of the system. We do this only to illustrate a potential constraint on the application of the
method for real systems; clearly results will depend on the particular parameters of the true system, which we do not explore in detail here. We will use our $10^6$ years of monthly values to calculate $E_{kkk}$, $A$, $G$ and $Q$ for different $10^2$ to $10^4$ years segments of data. Then we construct histograms of the values recovered and count the percentage of the time where the constraint (4.18) is met. This constraint is bound not to be fulfilled over some short segments of time, due to the inherent variability of a system forced by noise. Also, for short enough segments, we may not have the adequate data to recover matrix $M$ (4.11), so we will count those instances as well.

The first step to retrieve these parameters is to calculate $M_{kl} = A_{kl} + \frac{1}{2} E_{kkk}^2 \delta_{kl}$ from the data using:

$$M = \frac{1}{\tau} \log(C_\tau C_0^{-1})$$

where $C_0$ is the data covariance matrix and $C_\tau$ is the $\tau$ lag covariance matrix. This method is exactly the same used to retrieve $M$ in a regular LIM, and follows from (4.12). Although $M$ is independent of the lag used to calculate it, in practice some values of $\tau$ will produce better results than others. Good retrievals are found when using lags 5 to 30 months (lag 1 month retrievals are particularly bad). For the remaining of this chapter we will use a lag of 15 months to calculate $M$. Note that in practice, the reported success rate for calculating $M$ -or indeed the full CAM-LIM- may be larger than reported herein because in a specific case, a practitioner would attempt a retrieval with different lags, which we cannot do practically herein.

Figure 4.3 shows the histogram of the $A_{11}$ and $E_{111}$ retrieved values when using 500, 1000, and 10.000 years data segments. Similarly figure 4.4 shows the $G_{11}$, and $B_{11}$ histograms for the same intervals. For segments of data of this duration the retrievals are in general “good” with histograms centered more or less around the parameters “true” values (4.21). It is apparent also that there is great variability in the parameter estimations. For
example the value of \( E_{111} \) varies from 0.06 to 0.31, depending on what 500 years segment is used. The variability in the retrieval of \( E_{111} \) will covary with retrieval of other parameters in accord with (4.15, 4.16, 4.17). This shows that, although the system is stochastically generated by parameters satisfying the constraints in the system (4.18), attempts at recovering these parameters using shorter intervals may fail as (4.18) is not satisfied in that particular segment.

In this particular system, the retrieval of \( M \) fails 87.5\% (failing defined as not retrieving real values) of the time for 100 years segments. Out of that 87.5\%, the multilinear constraint (4.18) is met 70.5\% of the time. For 250 years \( M \) is retrieved successfully 98\% of the time, and the multilinear constraint is met 94.4\% of the time out of that 98\%. For 500 years segments virtually all retrievals are successful with 99.9\% \( M \) good retrievals, and 99.3\% out of that 99.9\% the constraint being satisfied. This has implications for interpreting the underlying dynamical process responsible for the skewness and kurtosis of real systems given the limited climatic record.

Similarly, figure 4.5 shows the histogram of the skewness and kurtosis of \( x_1 \) retrieved using 100, 1000, and 10000 years segments. The reason for the high rate of failure of estimating the stochastic parameters using 100 years segments is clear by looking at figure 4.5a. There are some non-negligible intervals of time where the system has actually the opposite skewness and have lighter than gaussian tails, clearly regions where (4.18) will not be satisfied. On the other hand, there are 100 years segments where an excess kurtosis of over 30 is computed. Each component of the system has heavier than Gaussian tails when considering segments of 1000 years or more. Even using such a long period of time excess kurtoses in excess of 15 are calculated over some rare, but non-negligible 1000 years spans.

The multilinear constraint (4.18) may be rewritten as:
\[ C(x_n) = \frac{3}{2} \sum_l M_{nl} S_{2n} S_{nn} - \sum_l M_{nl}(K_{ln} - 3V_{ln}) > 0 \] (4.23)

where we have defined the components of the \( K, V, \) and \( S \) matrices as

\[
K_{ln} = \frac{\langle x_l x_n^3 \rangle}{\langle x_n^2 \rangle^2}, \\
S_{ln} = \frac{\langle x_l x_n^2 \rangle}{\langle x_n^3 \rangle^{3/2}}, \hspace{1cm} V_{ln} = \frac{\langle x_l x_n \rangle}{\langle x_n^2 \rangle}, \] (4.24)

and \( S_{nn} \) is the skewness of variable \( x_n \). Figure 4.6 shows the constraint \( C(x_1) \) when calculating the statistics of the system over 100, 1000 and 10000 years segments. The constraint is met when \( C(x_1) > 0 \). For this particular variable the constraint is met 100\% of the time when using 10000 and 1000 years segments, but is violated 4.8\% of the time (in this particular \( 10^6 \) years integration) when calculating the statistics of the system over 100 years intervals. This implies that, even though a system may be completely described by this kind of framework, i.e. on the long term the constraints imposed on the system by the CAM-LIM underlying dynamics are perfectly met, in the short term there may be segments of data where the statistics of the system looks nothing like what we would expect theoretically. On real systems, this may imply misattributing deviations of Gaussianity to the wrong dynamical mechanism, due to lack of data.

4.5 Concluding Remarks

In this note we introduced a multilinear SGS driven by CAM noise, and a method to extract the CAM parameters from data. Compared to a regular LIM, this framework retains the same conditional evolution as well as covariance structure, while at the same time generating
skewness and kurtosis. This formulation establishes a very tight constraint on the statistical structure of the system generated, and it generalizes the 1-D CAM noise constraint $K \geq \frac{3}{2}S^2$ (Sardeshmukh and Sura 2009). This 1-D constraint has been shown to be relevant for different climate variables (Sardeshmukh and Sura 2009; Sardeshmukh and Penland 2015; Sardeshmukh et al. 2015). The multilinear constraint (4.18) (or (4.23)) is in general harder to calculate, and its relevance to observed variability will be a matter of future research.

We tested the retrieval method by using a 2 by 2 toy model. We generated 12 million months of synthetic data through the integration of the CAM-LIM toy model. We investigated the spread of the CAM-LIM parameter retrievals, as well as the spread of the skewness and kurtosis generated. As might be expected, the longer the segment, in general the better the retrieval. There is great variability in both the retrieved parameters, as well as skewness and kurtosis over “short” 100-500 year segments of the data. Interestingly, although the spread is greatly reduced, significant variability is observed even when dividing the synthetic data in 10000 year intervals. Finally, the shorter the segment used the more likely the sampled statistical structure will violate the constraint (4.18) (which is met by the ”true” statistics) imposed by the CAM-LIM dynamics. It is important to repeat that this toy model is not physically based and is used just for simplicity and expository purposes. As such the general lessons will remain valid when translating the CAM LIM use to real variables, but the specifics, such as the length of data needed for an accurate retrieval, will not.

Finally, a few comments on the practicality of this method when dealing with real climate variables. The multilinear CAM-noise dynamics imposes a hard constraint -(4.23) or (4.18)- on the statistics generated. Even when that constraint is met, the system may not appropriately close if (4.17) is below zero. This also has been shown to be an issue in the 1-D case (Sardeshmukh and Penland 2015). In addition to this, and given the short climate record, the amount of data needed may be prohibitive. One conclusion of this work
is that even if the variable in question does not meet (4.23) for a short sampling period (the length of the climate record), it does not necessarily imply that it was not generated by CAM-LIM dynamics. We suggest than rather than modeling the "true" statistics of the system, it is used to model the "observed" statistics. If sampled statistics adequately satisfy (4.23) the system can be modeled through said CAM-LIM noise paradigm. Although not true in all cases, generally a good indication that a variable will satisfy (4.23) is given by the 1-D version of this constraint (4.1). This method can be used to model the skewness, kurtosis, and the conditional spread of a variable, and it will be a good fit if the statistics are actually generated by a CAM noise process. But even if it is not, it will in general be a better alternative than just plain additive noise to model those features.
Table 4.1: Skewness and kurtosis of $10^6$ years joint data. The (add) legend denotes the joint $10^6$ years run under the regular LIM. The theoretical value for the skewness is 0 and kurtosis is 3 for both these variables. The (mult) legend denotes the joint $10^6$ years run under the CAM-LIM, and the (mult, 1-D) legend denotes the theoretical values for skewness and kurtosis for an uncoupled 1-D system with the same parameters as the CAM-LIM case (no cross terms) (Sardeshmukh and Penland 2015). All values given up to two significant figures.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$ (add)</td>
<td>0.00</td>
<td>3.00</td>
</tr>
<tr>
<td>$x_1$ (mult)</td>
<td>0.44</td>
<td>4.55</td>
</tr>
<tr>
<td>$x_1$ (mult, 1-D)</td>
<td>0.46</td>
<td>4.47</td>
</tr>
<tr>
<td>$x_2$ (add)</td>
<td>0.00</td>
<td>3.00</td>
</tr>
<tr>
<td>$x_2$ (mult)</td>
<td>-0.59</td>
<td>4.49</td>
</tr>
<tr>
<td>$x_2$ (mult, 1-D)</td>
<td>-0.64</td>
<td>4.90</td>
</tr>
</tbody>
</table>
Figure 4.1: a Monthly sea surface height skewness $S$ over the Tropical Pacific. b Monthly sea surface height excess kurtosis $K$ over the Tropical Pacific. c $K - \frac{3}{2}S^2$ over the region. All quantities calculated using the 1958-2011 time period ECMWF ORA-S4 data (Balmaseda et al. 2013).
Figure 4.2: \(a\) (c) \(x_1\) \((x_2)\) probability density function. \(b\) (d) \(x_1\) \((x_2)\) pdf negative tail. Positive tails behave similarly. The negative (positive) tail of the pdf under CAM-LIM is heavier than the additive case even when the pdf is positively (negatively) skewed, as shown in \(b\). The skewness and kurtosis of each variable are shown in table 4.1. The probability density functions were empirically fitted. Solid line denotes the additive case, and dashed line the CAM case.
Figure 4.3: **Left** Retrieval of $A_{11}$ calculated using segments of $a$ 500, $c$ 1000, and $e$ 10000 years. **Right** Retrieval of $E_{111}$ calculated using segments of $b$ 500, $d$ 1000, and $f$ 10000 years.
Figure 4.4: **Left** Retrieval of $G_{11}$ calculated using segments of a 500, c 1000, and e 10000 years. **Right** Retrieval of $B_{11}$ calculated using segments of b 500, d 1000, and f 10000 years.
Figure 4.5: **Left** (Right) Histogram of skewness (excess kurtosis) calculated over a (b) 100, c (d) 1000, and e (f) 10000 year segments of the $10^6$ years joint integration data. Notice that the horizontal axis scale shrinks from top to bottom.
Figure 4.6: Constraint $C(x_1) > 0$ for a 100 years, b 1000 years, and c 10000 years segments. Notice that the horizontal axis scale shrinks from top to bottom.
Chapter 5:

MODELING TROPICAL PACIFIC VARIABILITY THROUGH STATE DEPENDENT NOISE

In collaboration with Daniel J. Vimont, Cécile Penland, and Matthew Newman

Abstract

In this chapter we present an application of the method developed in Chapter 4 to model the tropical Pacific sea surface temperature (SST) statistical moments. From the available SST data we construct both a regular Linear Inverse Model (LIM) driven by additive Gaussian white noise, and a LIM driven by Correlated Additive-Multiplicative Gaussian white noise (CAM-LIM). As a consistency check it is shown that the parameters used to generate the CAM-LIM can correctly be retrieved from the algorithm shown in Chapter 4. After further modification of that algorithm it is found that the CAM-LIM correctly simulates the skewness and kurtosis of all but one principal component (PC), and as a consequence the skewness and kurtosis of Niño events are well described. This is in contrast to the Gaussian statistics generated by the regular LIM. Both the asymmetry and tails of the marginal and conditional probability density functions of the different PCs are better represented through this mechanism. The CAM-LIM maintains the same mean forecast equation as the regular
LIM, but it considerably modifies the forecast spread compared to the regular version. It is shown that, in contrast to the regular LIM, the CAM-LIM spread is in general asymmetric with respect to the mean forecast and it is also sensitive to the sign of the initial condition anomaly.
5.1 Introduction

The tropical Pacific variability is largely dominated by El Niño Southern Oscillation (ENSO) phenomenon. ENSO is a coupled ocean-atmosphere mode roughly defined as the quasi periodic warming of the tropical Pacific ocean at inter annual timescales. This warming is associated with a redistribution of the Tropical Pacific atmospheric mass. This process modifies the global circulation patterns, remotely modulating changes in precipitation, temperature (Halpert and Ropelewski 1992), hurricane activity (Smith et al. 2007), and many other processes, all over the world. There is an important degree of variability in the strength of this ENSO mode, with the biggest ENSO events having a total associated cost ranging in the billions of dollars (Nicholls 2001). For this reason parameterizations that improve the ENSO events distribution characterization are fundamental.

Important information about the distribution of ENSO events can be found through the calculation of statistical moments, –such as variance, skewness and kurtosis–, of the probability density functions of different ENSO indices. For example the most widely used ENSO index, the Niño3.4, is a positively skewed quantity with positive excess kurtosis (excess compared to the kurtosis of a Gaussian variable), as calculated through the 1854-2015 period. Those two simple measures gives us important information on the ENSO events distribution. From just these two measures, we know that through this period large events have usually been positive (Niños rather than Niñas), and that extreme events have occurred at a higher frequency than expected from a Gaussian variable. As positive Niño events are associated to roughly the opposite global effects compared to Niña events, and as extreme events have a much more expensive associated cost than their weaker counterparts, this is a very important piece of information that needs to be better represented in ENSO forecast models.

Different approaches have been used to explain ENSO deviations from Gaussianity, in-
cluding the non-linear wind stress response to convection anomalies (Kang and Kug 2002),
non linearity in the delayed thermocline feedback (DiNezio and Deser 2014), and other non
linear feedbacks (Jin et al. 2003; Liang et al. 2012). In contrast to these explanations, one of
the leading hypotheses put forward to explain these features is through stochastically gen-
erated systems (SGSs) driven by state dependent or multiplicative noise (Sura et al. 2005;
Sardeshmukh and Sura 2009; Levine and Jin 2015; Levine et al. 2016). In SGSs temporally
unresolved variability is stochastically parameterized (see Penland 1996, and Penland 2003
for more details on stochastic parameterization), with some of the stochastic terms being
dependent on the state of the system. In this paper we follow the parametrization scheme
laid out by Sardeshmukh and Sura 2009, under the approximations discussed in chapter 4.
In this scheme, the linearized deterministic low frequency dynamics of the system is driven
by correlated additive-multiplicative (CAM) Gaussian white noise, as well as some pure ad-
dditive noise. We have termed this kind of model a CAM Linear Inverse Model or CAM-LIM.
The amplitude of the multiplicative noise part depends linearly on the state of the system,
and the whole CAM noise amplitude is asymmetric with respect to the mean, i.e. some of
the additive noise is correlated to the multiplicative noise. This type of parametrization has
been shown to arise naturally in a quadratic non-linear system with clear separation between
slow and fast timescales (Sardeshmukh and Sura 2009; Sardeshmukh and Penland 2015).

It is instructive to contrast the differences between pure additive noise and CAM noise.
Pure additive noise acts completely independent from the system in question, that is the
distribution of the values taken by the noise is unaffected whether, for example, a strong el
Niño or a weak la Niña is occurring. On the other hand a noise parametrization that depends
on the state of the system, and that it is asymmetric with respect to the mean, accomplishes
two important things that contrast with the pure additive noise case: i) An initial noise
event modifies the probability and strength of the subsequent noise events, and ii) that
modification is sensitive to whether the system is in its positive or negative state. Those two conditions combined will imply a change on the skewness and kurtosis of the system with respect to the Gaussian case. Although any process that is fast (short decorrelation time scale), such as westerly wind bursts (Levine and Jin 2015), fast convective variations, rapid wind variability effect on surface fluxes (Sura and Newman 2008), etc, compared to the time scales of interests (e.g. monthly sea surface temperatures) is susceptible to being parameterized, at least approximately, in this way, we point out that we enter this analysis with no prior expectations on the system in question. All the relations will be derived from the data, and we will let the data dictate the values that our deterministic and stochastic operators will take.

In the previous chapter we developed a method to calculate the parameters of a CAM-LIM system based on the statistical structure generated by it. In this study we will parameterize the low frequency tropical Pacific sea surface temperature (SST) variability as a SGS driven by multivariate CAM noise, and we will contrast our results to the ones obtained using a regular LIM driven by additive noise (Penland and Sardeshmukh 1995). Of particular interest will be assessing changes in both the marginal and conditional PDFs arising from the noise state dependency, as well as estimating what part of the predictability of the tropical Pacific SSTs arises from the deterministic dynamics as opposed to the stochastic forcing. The remaining of the chapter is organized as follows: Section 2 will briefly review the model and data used. Section 3 will present and discuss the results, and Section 4 will present the conclusions.
5.2 Model and Data

As explained in Chapter 4, there is inherent variability in a system generated by noise. That variability increases in most cases if the noise depends on the state of the system. For this reason long segments of data are usually necessary in order to determine the system true statistics. In addition to this, the true statistical moments of a CAM-LIM system are tightly constrained (equation 4.23), and sampled statistics may not recover that constraint if the segment of data used to calculate it is too short. Due to the amount of data likely needed we have decided to construct both the regular LIM and the CAM-LIM using the NOAA Extended Reconstructed SST Version 3b (Smith and Reynolds 2003, 2004; Smith et al. 2008) monthly data covering 1854 to 2015. The grid resolution is 2 degrees longitude by 2 degrees latitude. The accuracy of the data decreases the further back in time we go (Smith and Reynolds 2004), but faced with the need of a long segment of data we have decided to keep the whole data set for the calculations. The motivation for this is practical, as the recent 60 years short segment satisfies fewer multivariate CAM-LIM constraints (4.23). It should be kept in mind that this will caveat the specific form of our results, but the value of the methodology presented will remain intact. For this reason a SST only LIM is calculated, as additional variables that usually complement LIMs (Newman et al. 2011) are not available over longer periods of time.

We calculate the empirical orthogonal functions (EOFs)/principal components (PCs) of the monthly SSTs over the [120E-70W, 20S-20N] region (with the 1854-2015 linear trend removed) and construct both our regular LIM and CAM-LIM in PC space keeping the first 15 PCs. All anomalies are calculated with respect to the 1854-2015 climatology. The variance explained by those 15 PCs amount to 91% of the total variance. For concreteness, figure 5.1 shows the first 3 EOFs (scaled by the square root of the variance explained by
each pattern) calculated throughout this period. EOF 1 is very similar to the familiar ENSO pattern, EOF 2 has some semblance to the more familiar EOF 2 calculated using more recent data (see Vimont et al. 2014 for a comparison), and it includes anomalies in the Central Pacific sandwiched by opposite sign anomalies in the eastern and Western Pacific. EOF 3 is characterized by anomalies extending from Baja California to the Western Pacific with opposite anomalies in the eastern Pacific. A visual inspection suggests that EOF 2 and 3 describe the Pacific Meridional Mode (PMM) (Chiang and Vimont 2004) in this data set. In fact, the PMM SST index (Vimont 2015) correlation to a combination of $PC_2$ and $PC_3$ is 0.8 during the 1948-2015 time period (1948 is the first year the PMM index is available).

Denoting the amplitudes of the principal components as $PC_i$, with $i = 1, 2, \ldots, 15$ the CAM-LIM equation for the evolution of $PC_i$ is given by

$$\frac{dPC_i}{dt} = \sum_j A_{ij}PC_j + \sum_k B_{ik}\eta_k + (G_i + E_iPC_i)\xi_i - \frac{1}{2}E_iG_i. \quad (5.1)$$

Here $A_{ij}$ is a stable linear operator that denotes the linearized deterministic dynamics, $B_{ik}$ denotes the amplitude of the pure additive noise, and the combination $G_i + E_iPC_i$ denotes the amplitude of the CAM noise. Notice that the CAM noise coefficients depend on the value of $PC_i$ at that particular time. More details about this model are given in chapter 4. All noise processes are independent of each other and have properties outlined in equation 1.5, (also see Penland 1996 and Penland 2003).

The parameters of the system (5.1) and its “true” statistics are related as follows (see
\[ E^2_i = -\sum_l M_{il} \frac{(2K_{li} - \frac{6}{\sigma_i^4} C_{li} - 3S_{li}S_{li})}{3(K_{li} - 1 - S_{li}^2)} \]  
(5.2)

\[ G_i = -\frac{\sigma_i}{2}(E_iS_{ii} + \sum_l \frac{M_{il}}{E_i} S_{li}) \]  
(5.3)

\[ Q_{ij} = -\sum_k (M_{ik}C_{kj} + C_{ik}M_{jk}) - G_i^2\delta_{ij} - E_i^2\sigma_i^2\delta_{ij} \]  
(5.4)

where \( \delta_{ij} \) is the Kronecker delta, the matrix \( Q = BB^T \), and the remaining quantities are defined as (with the symbol \(< >\) denoting long term averages)

\[ M_{il} = A_{il} + \frac{1}{2} E_i^2\delta_{il} \]  
(5.5)

\[ \sigma_i = <PC_i^2>^{1/2} \]

\[ C_{li} = <PC_lPC_i> = \sigma_i^2\delta_{li} \]

\[ S_{li} = \frac{1}{\sigma_i^2} <PC_lPC_i^2> \]

\[ K_{li} = \frac{1}{\sigma_i^2} <PC_lPC_i^3> . \]  
(5.6)

It is instructive to compare the CAM-LIM to an equivalent regular LIM driven by additive Gaussian white noise given below (compare to equation 5.1)

\[ \frac{dPC_i}{dt} = \sum_j M_{ij}PC_j + \sum_k b_{ik}\zeta_k, \]  
(5.7)

where in order to describe the same data \( M \) is equal to (5.5). In this equation \( \zeta_k \) denotes independent Gaussian white noise processes, and to avoid confusion with the CAM-LIM case the noise amplitudes \( b_{ik} \) are denoted uncapsitalized. The noise coefficients are related to the
statistics of the system as follows:

\[ q_{ij} = - \sum_k (M_{ik} C_{kj} + C_{ik} M_{jk}) \]  

(5.8)

where \( q = bb^T \), and the requirement that both systems generate the same data covariance structure implies that

\[ q_{ij} = Q_{ij} + G_i^2 \delta_{ij} + E_i^2 \sigma_i^2 \delta_{ij}. \]  

(5.9)

If one does not have a prior expectation of the system to describe, the natural first step is to calculate a regular LIM from the data. In that case one calculates \( M \) (equation 4.22) and calculates the noise coefficients by the use of (5.8). Equation (5.5) shows that more generally \( M \) may have contributions from both the deterministic dynamics (what is called the deterministic drift) encapsulated in \( A \), as well as a contribution from the state dependent noise (what is called the noise-induced drift, see Penland and Matrosova 1994 for a lengthier discussion). In the CAM-LIM case, by the procedure explained in chapter 4 we can make the distinction between those two contributions, while in the regular LIM case those contributions are always combined. Notice that if \( E = 0 \) in the CAM-LIM (this also implies that \( G = 0 \)), then \( B = b \) and the CAM-LIM reduces to the regular LIM. In that case \( M \) is simply \( A \).

Chapter 4 shows (equation 4.23) that the statistics of a system driven by this particular form of CAM noise are constrained in a distinctive way. Real data of finite length may or may not satisfy those constraints. There are several reasons why these constraints may not be satisfied. The most obvious one is that the underlying physical process generating the statistics is either not a CAM noise process (for example a deterministic non-linearity), it is not this form of CAM noise, or there are other important processes working together with CAM noise in generating the statistics. In addition or in contrast, as shown in Chapter 4 in
a model with perfect data, the constraint may not be satisfied simply because the amount of data needed to reliably calculate the statistics is more than the amount of data available. Here we will take the path of calculating the parametrization and evaluate to what point ENSO may be parameterized by this simple form of CAM noise. That is we will answer the question: Can ENSO be parameterized as a stable deterministic linear system driven by CAM noise?

5.3 Stochastic Modeling of Tropical SSTs statistical moments

Using the procedure outlined above (equations 5.2, 5.3, 5.4) and in chapter 4 (equations 4.15, 4.16, 4.17) we calculate the parameters of the CAM-LIM based on the available SST observations. The algorithm starts with the calculation of $M$ by using equation 4.22. Best results are obtained when $M$ is calculated using a lag of two months. In this case $E^2_i$ is positive for every PC, except PC 3. For that particular PC we increase the value of the input kurtosis $K_{33}$ in (5.2) to the point that the constraint is satisfied. Then after the CAM-LIM parameters are calculated we run both the CAM-LIM (5.1) and regular LIM 20 times, for 5000 years each time, and then blend the data. In both cases it is found that keeping or discarding the first 100 to 1000 years of each integration as a spin up time does not change the results, so we keep the whole 5000 years of each integration. In each case we have 100,000 years of data. We use the Heun integration scheme (Rüemelin 1982) for both cases, with a time step of three hours and collect monthly averages (1 month is equal to 30 days). In the regular LIM case the noise parametrization is such that (5.8) is satisfied.

Figure 5.2 shows the values of each PC variance ($\sigma^2_i$), skewness ($S_{ii}$), and excess kurtosis ($K_{ii} - 3$) as a result of the CAM-LIM and regular LIM integrations, as well as the observed values. The model statistics are calculated using the whole 100,000 years of data in each case,
and the observed values are calculated using just the 162 years of monthly data available from 1854 to 2015. As hoped, both the CAM-LIM and regular LIM reproduce the variance structure of the system. The skewness and excess kurtosis reproduced on the long term by the regular LIM correspond to values expected of Gaussian variables (zero skewness and zero excess kurtosis). There is a good measure of variability in those values if the statistics are calculated over shorter intervals (further discussed below). On the other hand the CAM-LIM model does produce deviations from Gaussianity in its statistics. Bearing in mind that the constraint (equation 4.23) was not satisfied for $PC_3$, it is surprising that actually the worst result was the modeling of $PC_2$. In general the CAM-LIM model does a good job of reproducing the skewness of the system, and barring $PC_2$ and $PC_3$, also the kurtosis of the system. Given that $PC_1$ is the principal component most associated with ENSO it is encouraging that this initial estimation does a very good job at reproducing the skewness and kurtosis of that PC.

Although the initial estimation provides good results, one may wonder if we can improve our parameterization such as to better reproduce the skewness and kurtosis of each PC. The answer is yes. Looking at the derivation of equations 5.2, 5.3, and 5.4 in chapter 4 (equations 4.15, 4.16, 4.17), we observe that there are several degrees of freedom that we could use to improve our parameter estimations –i.e. there are several combinations of the moments statistics that yield the same $E_i$ and $G_i$ values–. Our simple diagonal CAM-noise parametrization receives, as input, higher order cross values of the statistics, represented by the $S_{li}$ and $K_{li}$ matrices, with $l \neq i$. The simple parametrization at hand cannot fix those values (we would need off-diagonal CAM noise parameters to do it). Notice that our parameterization accounts for the covariances (in principle) of the system –through (5.4) and (5.8)–, but it does not account for the coskewness and cokurtosis. Rather than introducing off-diagonal CAM noise parameters (which is increasingly more complicated to do), we will
exploit the additional degrees of freedom of the system in the following way. With the CAM-LIM statistics generated, we will proceed to recalculate the parameters using (5.2), (5.3), and (5.4), re-inputing the observed variances, skewness, and kurtosis of the system, but leaving the off-diagonal values unchanged. With the parameters rederived, we repeat the 100,000 years total integration, recalculate the statistics generated, and then we repeat the procedure one more time (it is found that the system cannot be improved much after that). In each of those time steps the kurtosis and skewness of $PC_3$ are adjusted so the constraint given by (5.2) (or equation 4.23) is met. All the subsequent analysis and plots will refer to this “iterated estimation”.

After having calculated the new $E_i$, $G_i$, and $B_{ik}$ parameters, we run the CAM-LIM (5.1) once more, this time 20 times of 20,000 years each. We similarly run the regular LIM (5.7). Figure 5.3 shows each PC variance, skewness, and excess kurtosis calculated over the whole dataset length when using the new parameters (shown in figure 5.6). Compared to figure 5.2 we observe that the skewness and kurtosis modeled have improved considerably. Out of all results, the CAM-LIM modeled $PC_3$ kurtosis (and $PC_3$ skewness, though less pronounced) is found the be the worst match to the observed value, which is expected given that the constraint (4.23) was not met for that particular variable. Barring that PC, all the other estimations are remarkably good.

Although in the long term the model does a very good job at reproducing the statistical moments of the system, there is a great deal of variability that arises from its stochastic input. Figure 5.4 expands on figure 5.3b and 5.3c. For this figure, we calculate the skewness and excess kurtosis of the system over 162 years (the length of the observational dataset) segments. Plotted is the mean of the values obtained, as well as error bars indicating the 95% confidence interval in each case. The confidence interval is calculated by empirically estimating (using a Gaussian kernel) the distribution of skewness and kurtosis from the 2469
values obtained by subdividing the 400,000 years of CAM-LIM and regular LIM data in segments of 162 years (the first 22 years are discarded). There are several interesting aspects worth mentioning. First, the observed skewness falls outside the 95% confidence interval generated by the regular LIM for PCs 1, 3, 5, 8, and 12. Similarly the observed kurtosis falls outside the regular LIM 95% confidence interval for PCs 8 and 15. PC 1 kurtosis is barely inside that interval. This indicates that an approach that expands beyond the regular LIM is required to explain these deviations from Gaussianity. Second, in the CAM-LIM approach regimes that would be considered quite rare in the Gaussian world become more prevalent here. Third, there is a sizable number of 162 year segments where the PC distributions exhibit lighter than Gaussian tails. As discussed in chapter 4, in that regime the CAM-LIM constraints are not fulfilled in most cases, which underscores the fact that even 162 years of climate data is too short to calculate the stochastic CAM-LIM parameters. The fact that in this case 14 out of 15 constraints are satisfied is quite fortunate.

An important step to check the consistency of the model is to calculate whether we can retrieve our input parameters (5.2, 5.3, 5.4) through the statistics generated by the integration of (5.1). Figure 5.5a shows the input and figure 5.5b shows the CAM-LIM retrieved matrix $M_{ij} = A_{ij} + \frac{1}{2}E_i^2 \delta_{ij}$. In addition, figure 5.6 shows the input and retrieved $B_{ij}$, $G_i$, and $E_i$ calculated using the whole generated data set. To assess the relative importance of the pure additive noise compared to the CAM noise, all panels in figure 5.6 are plotted to scale, and $E_i$ is multiplied by a “typical” value of $PC_i$ provided by the respective PC standard deviation. Both figures 5.5 and 5.6 show that the correspondence between the input and retrieved parameters is good, which is an important check on the consistency of the model. There are several interesting points arising from figure 5.6. The first one is that for several PCs (roughly PC 4 and higher), the bulk of the stochastic forcing is given by the pure additive noise. Second, the $E_iPC_i$ magnitude is only moderate (less than $G_i$ and $B_{ij}$).
for small values of $PC_i$ and its effect is mostly felt when a relatively stronger event is taking place. Finally, the pure additive noise matrix amplitude is close to diagonal, which indicates that a diagonal CAM noise is actually a very reasonable approximation.

The differences in the stochastic parameterization between the CAM-LIM and regular LIM are illustrated in figure 5.7 for the stochastic forcing affecting the first PC. This figure shows the value of the input parameters $b_{1k}$ in the regular LIM case (5.7) and the pure additive $B_{1k}$ (upper panel) and CAM noise ($G_1 + E_1 PC_1$) in the CAM-LIM case (5.1). We plot the magnitude of the CAM noise coefficient rather than its signed value, as this coefficient multiplies a white noise process (see Penland 2003 for more details) that follows a Gaussian distribution, i.e. it takes positive or negative values with the same frequency over the long term. In the additive case, the noise forcing (averaged over a long span) is the same regardless of where in the system trajectory the system is (e.g. whether the system is in a Niña state or Niño state). Also, there is one noise process that overwhelmingly affects $PC_1$ compared to the others (the one denoted as $\zeta_1$ in equation 5.7). In the CAM-LIM case the off-diagonal pure additive noise structure is very similar to $b_{1k}$, but the $B_{11}$ coefficient is reduced compared to $b_{11}$ in order to account for the CAM noise amplitude variance. The CAM noise coefficient is asymmetric with respect to the mean (the CAM noise coefficient is zero when $PC_i = -\frac{G_i}{E_i}$). For example, its magnitude is equal to 5.33 when $PC_1$ has a positive value equal to 2 times its standard deviation (a major Niño is taking place), and it is almost zero when the standardized $PC_1 = -2$ (a major La Niña is taking place). So, for this particular PC, the CAM noise is very important for major El Niños (compare its value to the pure additive noise coefficient), and second order for La Niñas. This asymmetry translates into a positive skewness of this particular PC. Similarly, this parameterization generates excess kurtosis by the following reasoning. Imagine that $PC_1$ is in its neutral state. In that case, the CAM noise coefficient value is equal to 2.93 (the value of $G_1$).
Suppose that a positive noise event pushes the system to $PC_1 = 1$. In that case, the CAM noise coefficient increases to 4.13. This implies that the chances of a subsequent noise event that pushes the system even more out of equilibrium increase, and that the subsequent noise event is likely to be stronger than the previous one. At the timescales of interest, the SSTs see the noise as random, the subsequent event after the system was pushed to $PC_1 = 1$ may be negative, which may shut off the initial noise disturbance. Or it may be positive, which will further increase the likelihood of an extreme event. This shows that the system will make incursions into extreme values more frequently than if there were no state dependency. In this way, the kurtosis of the system is modified.

The process described above produces a measurable average effect in the trajectory of the system, “the noise-induced drift” that, together with the deterministic “drift”, accounts for the system predictability. The expectation value of the pattern at time $t$ (forecast) given the PC values at time 0 is given in both the CAM-LIM and regular LIM by (see equations 4.6 and 4.12):

$$PC_i(t) = G_{ij}(t)PC_j(0)$$

(5.10)

where

$$G(t) = \exp(Mt).$$

(5.11)

The optimal initial condition $\bar{X}_t(0)$ (in PC space) that maximizes growth at time $t$ satisfies (equations 2.8 or F.2):

$$G^T(t)G(t)\bar{X}_t(0) = \lambda_t \bar{X}_t(0),$$

(5.12)

where $\lambda_t$ indicates the variance growth from 0 to $t$ associated with the evolution from $\bar{X}_t(0)$ to $\bar{X}_t(t)$ through predictable dynamics. The collection of $\lambda_t$ values is known as the maximum amplification curve (Penland and Sardeshmukh 1995).

Our framework gives us a means to separate the noise (the $\frac{1}{2}E_i^2$ part in equation 5.5)
and deterministic drifts (the $A$ part in equation 5.5) in $M$. Figure 5.8a shows the 6 months optimal initial condition that maximizes SST variance, calculated using (5.12), and figure 5.8b shows its 6 months evolution. As expected, the optimal pattern evolves into a full fledged ENSO event (very similar to EOF 1). Figure 5.8c shows the maximum amplification curve when we use the whole $M$ operator in its calculation (deterministic + noise induced drift), as well as the same curve with no noise induced drift ($M = A$). Both curves look very similar with only a small increase in the growth when considering the noise induced drift. This indicates that the noise induced drift plays a very small role in predictability, and that the predictable dynamics is almost entirely determined by the deterministic part of the system. The optimal condition (figure 5.8a) and its evolution (figure 5.8b) are virtually unchanged whether or not we use the noise induced part in their determination. The $E$ operator enters the system mostly in the unpredictable (with respect to our timescales of interest) noise part. In hindsight, this result is expected, since the LIM framework is generally successful in its forecast of tropical SST anomalies (Newman and Sardeshmukh 2016). This implies that although important, the noise (of which the $E$ operator partially controls its amplitude) does not overwhelm the system. As a consequence, the value of $E$ is relatively small. In other less predictable contexts, it is reasonable to expect a bigger role of $E$ in accounting for a system predictability.

5.3.1 Modeling the events distribution

The pair $EOF_1$ and $PC_1$ explains most of the variance of the monthly tropical Pacific SSTs. As an expository point, we show the observed, regular LIM, and CAM-LIM modeled standardized $PC_1$ pdf in figure 5.9. The positive and negative tails of the distribution are explicitly shown in 5.9b and 5.9c. Complementary to this figure, table 5.1 states the pdf cumulative and exceedance values for some selected thresholds. This is done in order to
highlight the deviations from Gaussianity of the CAM-LIM model as well as the observations. It is not visually obvious from figure 5.9a, but the $PC_1$ values are asymmetrically distributed between positive and negative values, with 52% monthly values being negative. As expected, the regular LIM produces a Gaussian distribution of anomalies (figure 5.9) with 50% of the anomalies being negative and 50% positive. The CAM-LIM model does a noticeably better job at reproducing this asymmetry, as well as the other cumulative and exceedance probabilities shown in table 5.1, with just one exception (the $p(x \leq -1)$ value). Both figures 5.9b and 5.9c as well as table 5.1 show that the CAM-LIM also models the behavior of the tails noticeably better, with the CAM-LIM model more closely predicting the relative frequency of extreme positive and negative events. Although the results are very encouraging, the fit to observations is by no means perfect, since that would require the knowledge of all the statistical moments of the pdf, not only the variance, skewness and kurtosis. The most notable difference is that the most frequent value of the CAM-LIM generated pdf is slightly negative, whereas, in observations, that value is noticeable closer to the mean of the distribution. The asymmetry of the observed distribution is most recognizable in the difference between weak positive and negative anomalies (contrast the observed PDF values between 0 and -0.5, and 0 and 0.5). It is also conceivable that as observations keep accumulating some of the bumps shown in the observed tails (most prominently in figure 5.9c) will soften and the CAM-LIM and observed tails will look more alike.

Not only can we use the CAM-LIM to model the marginal pdf of the system, but also we can use it to model the conditional pdf, and as a result the expected spread of forecast values given an initial condition $PC_i(0)$. In specific, both the CAM-LIM and regular LIM produce the same forecast (see equations 4.6 and 4.12). On the other hand, it is reasonable to think that the spread, given by the unresolved variability, around that forecast will be different depending on whether we are using the regular LIM or CAM-LIM. For example, in
the regular LIM case, one would expect a symmetric spread around the forecast value. In addition to that, we would also expect the spread to be insensitive to the sign of the initial anomaly.

Although the CAM-LIM does not modify the SST prediction scheme (5.10), it has the potential to give us valuable information on the expected spread of a forecast. To exemplify this we go back in time to the tropical SST conditions in June 1997. To measure the evolution of the system we will use the Niño3.4 index as given by the first 2 PCs of the system as follows: $\text{Niño3.4} = \alpha_1 PC_1 + \alpha_2 PC_2$, where $\alpha_1 = 0.0493$ and $\alpha_2 = 0.0250$ are calculated using multivariate linear regression (this estimation correlation with the directly calculated index is 0.97). To illustrate the differences between the regular LIM and CAM-LIM, we use two initial conditions, the first one being the June 1997 conditions (representative of June 15, $\text{Niño3.4} = 1.43$), and its negative mirror image ($\text{Niño3.4} = -1.43$). We integrate these initial conditions 100,000 different times using the regular LIM and CAM-LIM setups for 6 months (June 15, 1997 to Dec 15, 1997) using a time step of 3 hours and we collect monthly means (that we could think of as monthly values centered on Jul 1, Aug 1, Sep 1, Oct 1, Nov 1, and Dec 1). We fit the 100,000 year values of the December 1 spread onto empirically determined pdfs (using a Gaussian kernel), resulting in the pdfs shown in figure 5.10. Conditional pdfs were also calculated for the Jul 1 to Nov 1 values given the Jun conditions (not shown), and expectedly show a progressive widening of the forecast spread. Returning to figure 5.10a, both regular LIM and CAM-LIM generated distributions have on December 1 the same average equal to the “forecast” given by (5.10) ($\text{Niño3.4} = 1.65$), but the respective forecast spread differ dramatically. The regular LIM generated pdf is very close to its theoretical Gaussian distribution (skewness of zero and kurtosis of 2.99), while the conditional CAM-LIM generated distribution is skewed to positive values (skewness of 0.44) and has noticeably more pronounced tails (kurtosis of 3.34). How would these December
conditional pdfs change had the initial condition been the exact opposite of the June 1997 monthly anomalies? The answer to that is shown in 5.10b. In this case the regular LIM produces the exact negative mirror image to the Gaussian conditional pdf shown in figure 5.10a, so under a regular LIM world positive and negative SST anomalies behave symmetrically around 0. In the CAM-LIM case, the pdf is much more symmetric (skewness of only -0.04) than in the positive case, and the heaviness of the tails has been toned down (kurtosis of 3.12). Given the positive June 1997 initial condition, this noise parametrization is able to produce a wider range of outcomes, and is less predictable as a result than in the comparative negative case (figure 5.10b), ranging from a very extreme ENSO to a complete shut down of the event. This result shows that not only is the CAM-LIM framework able to convincingly reproduce the observed marginal statistics of the system, but it is also able to produce different conditional statistics depending on where the system is.

We explore this further in figure 5.11. This figure uses the same setup explained in the previous paragraph and shows the forecast of the system (equation 5.10, initialized on June 15), the actual observations, and the evolution of the 95% regular LIM and CAM-LIM confidence intervals. Values are given every 15 days, with the monthly values being representative of day 15 of that particular month, and the value at day 1 being the average of two consecutive months. In the marginal sense, the Niño event of 1997 was quite rare and extreme under both regular LIM and CAM-LIM paradigms, being in the 99.97% tail of the regular LIM generated distribution, and in the 99.75% tail in the CAM-LIM generated distribution. In the conditional sense, given an initial state of Niño3.4 = 1.43 already in June, the record peak values observed later in the year can be explained by noise excitation of the system. Although the observed values deviate clearly from the forecast, they are well within the 95% spread given by both regular LIM and CAM-LIM integrations. As it may be expected from the increased kurtosis as well as non-zero skewness of the CAM-LIM generated
data (figure 5.9) compared to the regular case, the spread of the CAM-LIM forecast is wider and asymmetric with respect to the forecast value, which is consistent with the skewness of the conditional pdf shown in figure 5.10.

5.4 Conclusions

In this study we applied the procedure described in chapter 4 to calculate the parameters of a Linear Inverse Model driven by Correlated Additive-Multiplicative Gaussian white noise to model the tropical Pacific SST variability in principal component space. Our objective is to maintain the regular Linear Inverse Model attractive features; i.e. maintain a good representation of the variance of the system, as well as retain its good forecast abilities (Newman and Sardeshmukh 2016), but at the same time obtain a better representation of the higher statistical moments of the system. These moments, specifically skewness and kurtosis, are important in describing the asymmetry between positive and negative ENSO events, as well as the relative occurrence of extreme cases. As such, modeling the right statistical output has very practical consequences. The CAM-LIM, as opposed to the Gaussian regular LIM output, clearly succeeds at representing the observed skewness and kurtosis of the system, with the exception of principal component 3.

Using this method, an explicit breakdown of the deterministic dynamics and noise induced drift contribution to the predictability of the system (encapsulated by matrix $M$ in the regular LIM case) can be calculated. Our results suggest that almost all the predictable signal arises through the deterministic linearized dynamics, and that the state dependent noise contribution to it is secondary. The main effect of the noise state dependency is to alter the statistics of the system through changing the underlying probability of noise events as a function of the system trajectory. That state dependency is in general asymmetric, and the
degree of asymmetry is calculated from the data through the $G$ computation. In absolute terms, most of the noise contribution is given by noise that is independent of the state of the system, with the CAM noise contribution being more important for the first few PCs, as well as when the system deviates considerably from its neutral state (compare the shadings in figure 5.6).

Not only can we use this framework to estimate the statistical moments of the marginal pdf of the different PCs, but also to study the conditional spread of the forecast, and conditional pdf given an initial SST anomaly. Both regular LIM and CAM-LIM generate the same forecast (see equations 4.6 and 4.12), but the spread of values predicted around that forecast vary noticeably depending on the LIM considered. A forecast is only perfect if the noise contribution somehow cancels along the system trajectory. In most cases the forecast will be modified by noise, and that modification will depend on the particular framework. In the regular LIM case the spread around the forecast is symmetric, while in the CAM-LIM case the positive and negative spread will depend on where the system is. A specific example of that was given when comparing the December spread of a forecast initialized with the monthly June 1997 SST anomalies, and its negative mirror image (see figure 5.10a, and 5.10b). In that case, a positive extreme event on December was more likely, for the positive June anomaly (fig 5.10a), than an extreme negative event for the negative June anomaly (fig 5.10b). The regular LIM cannot make such a distinction. Using the CAM-LIM framework, this type of analysis can be performed at almost real time and could be used as a standard tool to better assess the potential spread of a forecast.

There are several caveats that should be pointed out. Given the tight CAM-LIM constraints and the “short” climate record, we decided to calculate these models using just SST data. We hope to introduce some measure of the ocean heat content (Newman et al. 2011) to complement the SST data in a newer generation of CAM-LIM. This is not an easy task.
and will need some creative thinking to introduce this complementary information, while still satisfying the system constraints. Also, the use of data going as far back as 1854 is clearly non-satisfactory. A systematic study of how much of the data we can cut while still getting good results, as well as comparing this CAM-LIM to other versions generated using alternative SST data reconstructions, such as HadISST (Rayner et al. 2003), is in order. There are two important approximations used in the calculation of the CAM-LIM. Our results suggests that the first one –the noise is diagonal– is a rather good one, while the second –the CAM noise just depends on the state of the system it affects– remains untested. Relaxing that last approximation would introduce steep mathematical complications, but potentially may not only help explaining the system skewness and kurtosis, but also the higher moments cross-statistics as well (coskewness and cokurtosis). Despite these caveats, this stochastic parameterization is clearly a step forward to represent the important tropical Pacific variability.
Table 5.1: Important $x = \hat{P}C_1 = \frac{P_{C_1}}{\text{std}(P_{C_1})}$ probability cumulative and exceedance thresholds. For context, the maximum value of $\hat{P}C_1$ in this dataset is equal to 3.91 and was achieved on November 1997.

<table>
<thead>
<tr>
<th>Case</th>
<th>$p(x \geq 0)$</th>
<th>$p(x \geq 1)$</th>
<th>$p(x \leq -1)$</th>
<th>$p(x \geq 2)$</th>
<th>$p(x \leq -2)$</th>
<th>$p(x \geq 3)$</th>
<th>$p(x \leq -3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed</td>
<td>48.0%</td>
<td>15.3%</td>
<td>15.9%</td>
<td>3.5%</td>
<td>1.6%</td>
<td>0.78%</td>
<td>0.02%</td>
</tr>
<tr>
<td>CAM</td>
<td>47.5%</td>
<td>15.3%</td>
<td>15.2%</td>
<td>3.2%</td>
<td>1.4%</td>
<td>0.53%</td>
<td>0.04%</td>
</tr>
<tr>
<td>Additive</td>
<td>50.0%</td>
<td>15.9%</td>
<td>15.9%</td>
<td>2.3%</td>
<td>2.3%</td>
<td>0.13%</td>
<td>0.13%</td>
</tr>
</tbody>
</table>
Figure 5.1: (a) 20S-20N Sea surface temperature EOF 1 (1854-2015). (b) 20S-20N Sea surface temperature EOF 2 (1854-2015). (c) 20S-20N Sea surface temperature EOF 3 (1854-2015).
Figure 5.2: **Initial Estimation**  

- **a** Variance explained by each PC.  
- **b** Each PC skewness.  
- **c** Each PC excess kurtosis. Blue circles denote observed values, red plus signs CAM noise modeled values, and yellow crosses additive noise modeled values.
Figure 5.3: Iterated Estimation a Variance explained by each PC. b Each PC skewness. c Each PC excess kurtosis. Blue circles denote observed values, Red plus signs CAM noise modeled values, and yellow crosses additive noise modeled values.
Figure 5.4: **a** Mean and 95% confidence interval of different PC skewness calculated over 162 year segments **b** Mean and 95% confidence interval of different PC excess kurtosis calculated over 162 year segments. Red circles denote observed values, blue asterisks CAM noise modeled values, and pinks Xs additive noise modeled values.
Figure 5.5: a Input $M$ matrix. b Retrieved $M$ matrix, with entries given by $M_{ij} = A_{ij} + \frac{1}{2}E_i^2$, after 400,000 years of generated data using equation 4.22 and input parameters estimated by the method described in the text and shown in figure 5.3.
Figure 5.6: a Input B matrix. b Retrieved B matrix. c Input G matrix. d Retrieved G matrix. e Input $E_i \sigma_i$ matrix, where $\sigma_i$ is the standard deviation of $PC_i$ calculated from the available 1854-2015 data. f Retrieved $E_i \sigma_i$ matrix, where $\sigma_i$ is calculated from the 400,000 years CAM-LIM integration. All retrieved quantities are calculated using the 400,000 years of generated data using equations 5.2, 5.3, 5.4 and input parameters estimated by the method described in the text.
Figure 5.7: a Pure additive noise coefficients $B_{1k}$ affecting the development of PC 1 in the additive case (dashed red) and CAM case (blue solid). b CAM noise coefficient $(G_1 + \hat{E}_1 \hat{PC}_1$, where $\hat{E}_1 = \text{std}(PC_1)E_1$) absolute value as a function of the standardized $\hat{PC}_1 = \frac{PC_1}{\text{std}(PC_1)}$ value.
Figure 5.8: (a) SST 6 months optimal. (b) 6 months evolution. (c) Maximum amplification curve. Solid line represents the total deterministic + noise induced drift contribution to predictable growth. Dashed line represents only the deterministic contribution.
Figure 5.9: a Observed (black), additive noise modeled (blue), and CAM noise modeled (red) standardized PC 1 probability density functions. b Negative tail. c Positive tail.
Figure 5.10: **a** Spread of the Jun 1997 initial condition \((\text{Niño}3.4 = 1.43\) representative of June 15) when integrated using the regular LIM (add) and the CAM-LIM for 6 months. Shown is the distribution of the Nov 15 - Dec 15 (representative of Dec 1) monthly average of 100,000 different integrations. **b** As of **a**, but using the negative mirror image of the Jun 1997 conditions \((\text{Niño}3.4 = -1.43)\) as initial input.
Figure 5.11: June 1997 LIM forecast (blue line) enveloped by the 95% confidence interval generated by the regular LIM (light blue shading), and 95% confidence interval generated by the CAM-LIM (light red shading). This is contrasted with the actual observations (black line).
In this thesis we have looked at two different kinds of models that are relevant to understand the tropical climate variability. This work was divided in two, the first part (chapters 2 and 3) takes a deterministic and analytical look at the processes that thermally couple the ocean and atmosphere. Part two (chapters 4 and 5), on the other hand, deals with the stochastic treatment of tropical variability using inverse methods. We look at these two topics separately, although there is a potential overlap between these approaches that may be considered in future research endeavors (e.g. “Stochastic modeling of the WES feedback”).

In chapter 2, we used a simple linear model that portrays the WES feedback coupling and study how the SST-atmosphere coupled patterns vary under structural variations in the mean state. We consider coupling variations in the meridional direction ($y$) arising through the mean state meridional structure. In this model, the atmosphere to ocean coupling is controlled by parameter $\alpha(y)$ (the ocean latent heat sensitivity to zonal wind variations), and the ocean to atmosphere coupling is controlled by parameter $K_q(y)$ (the atmospheric convective heating sensitivity to SST variations). The effective coupling of the system is given by $K_q(y)\alpha(y)$, which implies that the location and overlap in the couplings are important elements to determine where the WES feedback can effectively grow ocean-atmosphere coupled structures. This effective coupling varies seasonally, and as a consequence, the structure of the modes that the coupling sustains varies with it. Variations in the effective coupling
are found to be mainly associated to changes in the structure of the ocean to atmosphere coupling (the $K_q(y)$ parameter). It was found that the mean state effectively acts as a “mode selector” (this is expanded upon in chapter 3) of the coupled variability. As such, the location and shape of the effective coupling greatly affects the structure and growth of the coupled modes. For example, it was found that a thin equatorially symmetric coupling window enhances only the symmetric response, and as a consequence, the variability remains tightly confined to the equatorial region. When the symmetric coupling is wider, the equatorially antisymmetric meridional mode-like structures start dominating the coupled response. This is traced back to differences in the atmospheric wave response to SST patterns and is the topic of chapter 3. We also investigated how the system responded to a coupling structure that is equatorially asymmetric. This is motivated by the asymmetry in the location of the ITCZ in both the tropical Atlantic and eastern Pacific, and also the equatorial asymmetry of the AMM in boreal fall. It is found that the resulting structures follow the coupling regions and as a result are asymmetric themselves, resembling the AMM behavior. In addition, transient growth is maximized when the coupling region is centered off-equator. The model is written in terms of a Gill-like atmospheric response, but it can be easily applied to model a boundary layer atmospheric response to SST variations.

Chapter 3 takes an analytical look at the processes that contribute to growth and propagation through the WES feedback. Under some approximations, the atmospheric response was decomposed in terms of different SST $T_m$ modes (by projecting onto different meridional parabolic cylinder functions $\psi_m(y)$). Under a homogeneous coupling parameterization the variability naturally decomposes onto an equatorially symmetric and an equatorially antisymmetric response. The governing equations have the same shape in both cases except close to the near equatorial region. In this case the atmospheric Kelvin wave acts in most of the parameter space to damp the symmetric response, and the first atmospheric antisymmetric
Rossby wave acts to strengthen the antisymmetric response. This explains why equatorially antisymmetric modes are preferentially excited by the WES feedback. However, there is an important region in the parameter space where the symmetric response is important and should be considered when interpreting the observed variability sustained by the WES feedback. Several sensitivity studies were conducted further reinforcing the previous conclusions. The growth process is shown to be fundamentally non-normal, with a distinctive propagation from high SST $m$ modes to lower $m$ modes. This translates into an equatorward movement of the structure in physical space. The non-normality manifests itself as interactions that initially grow the SST anomaly (specifically the interaction of Rossby waves $m+1$ and $m-1$ with mode $T_m$) while at the same time seeding its eventual decay. This is further demonstrated by artificially suppressing the non-normality and then comparing the responses. Several relevant mathematical formulas (most of them in the appendix sections) that relate each mode magnitude and phasing with the SST variance growth are laid out.

Chapters 4 and 5 target a different issue. In this case, we studied the mechanisms to generate skewness and kurtosis in a linear inverse model framework. In a regular LIM, additive state independent stochastic forcing drives the low-frequency dynamics encapsulated in a stable linear operator. This framework has been very successful in forecasting tropical Pacific SST anomalies (Newman and Sardeshmukh 2016) and has also been used to study the ENSO relevant physics (Penland and Sardeshmukh 1995; Penland 1996; Newman et al. 2011; Vimont et al. 2014). In chapter 4, we developed a stochastic parameterization that goes beyond the additive noise parameterization used in a regular LIM. In this case we also consider the effect of fast variability that depends on the state of the system, or multiplicative noise. In addition we allow some of the additive noise to be correlated with the multiplicative noise (this arises naturally in a linearization of a quadratic flux with separation between slow and fast timescales (Sardeshmukh and Sura 2009; Sardeshmukh and Penland 2015)) and we
therefore obtain a stochastically generated system that can account for the observed skewness and kurtosis (as well as variance). In this chapter we calculated the relation between the parameters of this CAM-LIM model and the long term statistics generated by it. We use this relationship to calculate those parameters from data. This is illustrated in a perfect data setup in a 2 by 2 toy model. Chapter 5 uses this algorithm (with some minor modifications) to generate a linear model driven by correlated additive-multiplicative noise that can account for most of the observed variance, skewness, and kurtosis of the Tropical Pacific SSTs. The model reproduces all but one principal component ($PC_3$) skewness and kurtosis. In addition to better modeling the higher order PC statistics compared to the regular LIM, the CAM-LIM model also performs better in modeling the conditional forecast spread. Finally, an explicit breakdown of the system predictable part in terms of the deterministic dynamics and state dependent noise can be calculated using this framework. The results suggest that the predictable growth of the system is mostly controlled by the deterministic dynamics.
Appendix A:

Let us reorder the state vector as $\Phi = (T, u, v, \phi)^T$. In this way the matrix $M$ for an individual point $y_i$ is

$$
M = \begin{pmatrix}
-\epsilon_T + \gamma \nabla^2 & \alpha(y) & 0 & 0 \\
0 & -\epsilon & y & -ik \\
0 & -y & -\epsilon & -\frac{\partial}{\partial y} \\
K_q(y) & -ik & -\frac{\partial}{\partial y} & -\epsilon.
\end{pmatrix}
$$

(A.1)

The symbol $\frac{\partial}{\partial y}$ is a numerical derivative, and connects this block to similar blocks acting on $y_{i-1}$ and $y_{i+1}$.

To find the spectra $\{\Phi_j, -i\omega_j\}$ of $M$ we can solve the determinant $|M + i\omega_j 1| = 0$. That is

$$
\begin{vmatrix}
-\epsilon_T + \gamma \nabla^2 + i\omega_j & \alpha(y) & 0 & 0 \\
0 & -\epsilon + i\omega_j & y & -ik \\
0 & -y & -\epsilon + i\omega_j & -\frac{\partial}{\partial y} \\
K_q(y) & -ik & -\frac{\partial}{\partial y} & -\epsilon + i\omega_j
\end{vmatrix} = 0
$$

(A.2)

We can separate this in an uncoupled part (Matsuno atmospheric model Matsuno (1966)) and a part that contains the coupling
\[
(-\epsilon_T + \gamma \nabla^2 + i \omega_j)
\]

We can further decompose the last determinant as

\[
\begin{vmatrix}
\begin{array}{ccc}
-\epsilon + i \omega_j & y & -ik \\
-y & -\epsilon + i \omega_j & -\frac{\partial}{\partial y} \\
-ik & -\frac{\partial}{\partial y} & -\epsilon + i \omega_j
\end{array}
\end{vmatrix}
- \alpha
\begin{vmatrix}
\begin{array}{ccc}
0 & y & -ik \\
0 & -\epsilon + i \omega_j & -\frac{\partial}{\partial y} \\
-K_q & -\frac{\partial}{\partial y} & -\epsilon + i \omega_j
\end{array}
\end{vmatrix} = 0
\] (A.3)

This shows that the coupled part of the system depends on the product \(K_q(y)\alpha(y)\). If the system were uncoupled \((K_q(y)\alpha(y) = 0)\), the eigenvalues would just be the ones given by the Matsuno atmospheric model (first determinant), plus another one equal to the oceanic damping. We further notice that the Matsuno model is normal, so the source of non normality comes from the effective coupling.
Appendix B:

Any optimal can be decomposed into an equatorially symmetric part ($\frac{1}{2}p_{sym}$) and an anti-symmetric part ($\frac{1}{2}p_{ant}$). As a consequence we can write an optimal constrained to start in the, say NH, $p_{NH}$ as

$$p_{NH} = \frac{1}{2}p_{sym} + \frac{1}{2}p_{ant}, \quad (B.1)$$

where $p_{sym}$ has the same amplitude as $p_{ant}$ (and $p_{NH}$) in the NH.

For a symmetric mean state both $p_{sym}$ and $p_{ant}$ will evolve conserving their symmetry. In this case, applying $G_i^\dagger G_r$ (see eqn 2.8) to the previous equation we find that either $p_{sym}$ or $p_{ant}$ is equal to the leading unconstrained optimal $p$. In this way we can write

$$p_{NH} = \frac{1}{2}p + \frac{1}{2}\hat{p} \quad (B.2)$$

where $\hat{p}$ is the opposite symmetry structure such that it cancels $p$ contribution in the SH. This implies that the constrained optimal will just be "half" the unconstrained optimal.

Let’s take for example figure 2.3. In this case figure 2.3c correspond to $p_{NH}$, the unconstrained optimal $p$ correspond to $p_{ant}$ (figure 2.3a) and $\hat{p} = p_{sym}$. Similar idea holds (approximately) for figure 2.4, but in this case $p = p_{sym}$ and $\hat{p} = p_{ant}$. 
Appendix C:

The version of the parabolic cylinder functions used in this study are given by:

$$\psi_m(y) = \frac{1}{\sqrt{2m!\pi^{1/2}}} H_m e^{x(-y^2/2)} \quad (C.1)$$

where $H_m$ are the physicist Hermite polynomials. These polynomials are defined for non-negative $m$ integers. The first 4 functions are plotted in Fig. C.1. The functions are orthonormal

$$\int_{-\infty}^{\infty} \psi_m(y) \psi_n(y) dy = \delta_{mn}. \quad (C.2)$$
Figure C.1: First four parabolic cylinder functions.
Appendix D:

We write the atmospheric variables using the parabolic cylinder functions as

\[
\begin{align*}
    u(t, x, y) &= \sum_{m=0}^{\infty} u_m(t) \psi_m(y) \exp(ikx) \\
    v(t, x, y) &= \sum_{m=0}^{\infty} v_m(t) \psi_m(y) \exp(ikx) \\
    \phi(t, x, y) &= \sum_{m=0}^{\infty} \phi_m(t) \psi_m(y) \exp(ikx)
\end{align*}
\] (D.1)

In solving the model equations for a particular mode \( m \) the following identity is useful:

\[
\begin{align*}
    y\psi_m &= \sqrt{\frac{m+1}{2}} \psi_{m+1} + \sqrt{\frac{m}{2}} \psi_{m-1} \\
    \frac{d\psi_m}{dy} &= -\sqrt{\frac{m+1}{2}} \psi_{m+1} + \sqrt{\frac{m}{2}} \psi_{m-1},
\end{align*}
\] (D.2)

as well as defining the auxiliary variables (Gill 1980):

\[
\begin{align*}
    q &= \phi + u \\
    r &= \phi - u
\end{align*}
\] (D.3)

Plugging equation D.1 into the model equations (3.1), and using equations 3.2, D.2 and
D.3 we find that

\[ q_0 = -K_q \frac{T_0}{\epsilon + ik} \]
\[ q_{m+1} = K_q \left( \frac{\sqrt{m(m+1)}T_{m-1} + mT_{m+1}}{-(2m+1)\epsilon + ik} \right) \]  \hspace{1cm} (D.4)

in doing so we used that

\[ r_{m-1} = \sqrt{\frac{m+1}{m}} q_{m+1}, \]  \hspace{1cm} (D.5)

which is a consequence of making the long-wave approximation (Gill 1980) in the meridional wind equation. Inverting equation D.3 and using equation D.5 we get

\[ u(t, y, x) = \frac{1}{2} (q_0(t)\psi(y) + \sum_{m=1}^{\infty} q_{m+1}(t)(\psi_{m+1}(y) - \sqrt{\frac{m+1}{m}}\psi_{m-1}(y)))exp(ikx) \]
\[ \phi(t, y, x) = \frac{1}{2} (q_0(t)\psi(y) + \sum_{m=1}^{\infty} q_{m+1}(t)(\psi_{m+1}(y) + \sqrt{\frac{m+1}{m}}\psi_{m-1}(y)))exp(ikx). \]  \hspace{1cm} (D.6)

The meridional wind can be found by plugging the above equations into the zonal wind and geopotential equations and solving for \( v \) yielding:

\[ v(t, y, x) = \sum_{m=0}^{\infty} \sqrt{\frac{1}{2(m+1)}}(K_q T_{m+1}(t) + (\epsilon + ik)q_{m+1}(t))\psi_m(y)exp(ikx) \]  \hspace{1cm} (D.7)

Equations D.6 and D.7 correspond to equation 3.3 in the main text.
Appendix E: Growth equation

Using equations (3.2) and (D.1) we can write the equation for the evolution of $T_m$ and $T_m^*$ as

$$\frac{\partial T_m}{\partial t} = \alpha u_m - \epsilon_T T_m$$
$$\frac{\partial T_m^*}{\partial t} = \alpha u_m^* - \epsilon_T T_m^*$$  \hspace{1cm} (E.1)

Multiplying the first equation by $T_m^*$ and the second one by $T_m$ and adding them we get

$$\frac{\partial |T_m|^2}{\partial t} = \alpha (u_m T_m^* + u_m^* T_m) - 2\epsilon |T_m|^2.$$  \hspace{1cm} (E.2)

Using equations D.3, D.5 and D.4 we find that the growth of a particular mode is governed by

$$\frac{\partial |T_m|^2}{\partial t} = K_q \alpha \left[ -\frac{1}{((2m-1)^2 \epsilon^2 + k^2)} ((m-1)(2m-1)\epsilon) |T_m|^2 ight.$$  
$$+ \frac{1}{((2m+3)^2 \epsilon^2 + k^2)} ((m+2)(2m+3)\epsilon) |T_m|^2 
- \frac{1}{((2m-1)^2 \epsilon^2 + k^2)} ((m-1)(2m-1)\epsilon) |T_{m-2}| |T_m| \cos(\delta_m - \delta_{m-2}) 
+ \frac{1}{((2m+3)^2 \epsilon^2 + k^2)} ((m+2)(m+1)(2m+3)\epsilon) |T_m||T_{m+2}| \cos(\delta_m - \delta_{m+2}) 
- \frac{1}{((2m-1)^2 \epsilon^2 + k^2)} \sqrt{m(m-1)k} |T_m||T_{m-2}| \sin(\delta_m - \delta_{m-2}) 
+ \frac{1}{((2m+3)^2 \epsilon^2 + k^2)} \sqrt{(m+2)(m+1)k} |T_m||T_{m+2}| \sin(\delta_m - \delta_{m+2}) \right] - 2\epsilon_T |T_m|^2.$$  \hspace{1cm} (E.3)
Here $\delta_m - \delta_{m-2}$ is the phasing in the complex plane between modes $m$ and $m-2$, and comes from the decomposition $T_m = |T_m|\exp(i\delta_m)$.

To find the equation for total SST growth we use equation 3.2 and write the total SST amplitude squared $|T|^2$ in terms of the individual modes.

$$|T|^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} T^*_m T_n \psi_m \psi_n$$  \hspace{1cm} (E.4)

Integrating $|T|^2$ over the whole domain (the symbol $<>$ denotes that integration) we get the total growth that can be decomposed in terms of the growth of individual terms as follows

$$<|T|^2> = \int_{-\infty}^{\infty} \int_0^{L_x} |T|^2 \, dx \, dy = L_x \sum_{m=0}^{\infty} |T_m|^2,$$ \hspace{1cm} (E.5)

where $L_x$ is the zonal extension of the basin considered and we have used equation C.2.

Using equation E.3 the equation for the total SST growth is given by

$$\frac{\partial}{\partial t} <|T|^2> = L_x K' \alpha \sum_{m=0}^{\infty} \left[ \frac{1}{((2m-1)^2 \epsilon^2 + k^2)} \right] ((m-1)(2m-1)\epsilon)|T_m|^2$$

$$+ \left[ \frac{1}{((2m+3)^2 \epsilon^2 + k^2)} \right] ((m+2)(2m+3)\epsilon)|T_m|^2$$

$$+ \left[ \frac{2}{((2m+3)^2 \epsilon^2 + k^2)} \right] (\sqrt{(m+2)(m+1)}k)|T_m||T_{m+2}| \sin(\delta_m - \delta_{m+2})$$

$$- 2L_x \epsilon_T \sum_{m=0}^{\infty} |T_m|^2.$$ \hspace{1cm} (E.6)

Notice there is a partial cancellation between the exchange terms.
Appendix F: Growth optimization

To find the initial conditions that maximize SST growth by the WES feedback at some time later $\tau$ we use a standard Lagrangian multiplier approach. The function $\mathcal{L}_\tau$ contains the information to optimize.

$$\mathcal{L}_\tau = |T(\tau)|^2 - \lambda(|T(0)|^2 - c). \quad (F.1)$$

Here $T(\tau)$ corresponds to the vector $(T_0(\tau), T_1(\tau), T_2(\tau), \ldots)$. The first term in the right hand side $|T(\tau)|^2 = \sum_{m=0}^{\infty} |T_m(\tau)|^2$ (compare to equation E.5, we are optimizing $\frac{1}{L_x} < |T(\tau)|^2 >$) is the quantity to be optimized, while the second term is the constraint, i.e. initial condition amplitude equal to some arbitrary constant $c$. Using (5.3) and taking the derivative of the previous equation respect to $T(0)$, we find that the initial condition $T(0)$ that maximizes growth at time $\tau$ satisfies the following eigenvalue problem:

$$G^\dagger(\tau)G(\tau)T(0) = \lambda T(0), \quad (F.2)$$

where $\lambda$ corresponds to the growth at time $\tau$. 
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