An Analytical Framework for Understanding Tropical Meridional Modes

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ABSTRACT

A theoretical framework is developed for understanding the transient growth and propagation characteristics of thermodynamically coupled, meridional mode–like structures in the tropics. The model consists of a Gill–Matsuno-type steady atmosphere under the long-wave approximation coupled via a wind–evaporation–sea surface temperature (WES) feedback to a “slab” ocean model. When projected onto meridional basis functions for the atmosphere the system simplifies to a nonnormal set of equations that describes the evolution of individual sea surface temperature (SST) modes, with clean separation between equatorially symmetric and antisymmetric modes. The following major findings result from analysis of the system: 1) a transient growth process exists whereby specific SST modes propagate toward lower-order modes at the expense of the higher-order modes; 2) the same dynamical mechanisms govern the evolution of symmetric and antisymmetric SST modes except for the lowest-order wavenumber, where for symmetric structures the atmospheric Kelvin wave plays a critically different role in enhancing decay; and 3) the WES feedback is positive for all modes (with a maximum for the most equatorially confined antisymmetric structure) except for the most equatorially confined symmetric mode where the Kelvin wave generates a negative WES feedback. Taken together, these findings explain why equatorially antisymmetric “dipole”-like structures may dominate thermodynamically coupled ocean–atmosphere variability in the tropics. The role of non-normality and the role of realistic mean states in meridional mode variability are discussed.

1. Introduction

Distinct patterns of low-frequency tropical–subtropical ocean–atmosphere coupled variability in the Pacific and Atlantic occurs mainly via two distinct feedbacks: the so-called Bjerknes feedback [Bjerknes 1969; note, however, that the Bjerknes feedback is now understood to involve additional dynamical processes in the ocean (Trenberth et al. 1998)] and wind–evaporation–sea surface temperature (WES) feedback (Xie and Philander 1994; Chang et al. 1997). Both of these feedbacks imply a mutual reinforcement between winds and sea surface temperature (SST), but via fundamentally different mechanisms. The Bjerknes feedback relies on ocean dynamical processes in linking wind anomalies to SST tendencies, and typically results in coupled modes that are largely equatorially symmetric such as El Niño–Southern Oscillation (ENSO). Outside the equatorial zone but within the tropics surface heat fluxes, especially latent and shortwave, play an increasingly important role in SST variability. There, the WES feedback links surface winds to SST tendency through changes in evaporation rates: if positive SST-induced wind anomalies are directed opposite to the mean wind direction, the resulting decrease in total wind speed is associated with a reduction in evaporation, and hence a positive SST tendency. The feedback loop is completed in both cases by SST affecting the winds via deep convection (Gill 1980) or boundary layer processes (Lindzen and Nigam 1987; Battisti et al. 1999).

The WES feedback has mainly been associated with maintaining equatorially antisymmetric coupled modes (meridional modes; Servain et al. 1999; Chiang and Vimont 2004) by the following mechanism. Consider an anomalous equatorially antisymmetric SST dipole. The atmospheric response to such a dipole includes anomalous winds blowing from the cold hemisphere into the warm hemisphere. In part because of the Coriolis effect, these anomalous winds relax the mean easterly trades in the warm hemisphere and strengthen them in the cold hemisphere. By wind modulation of evaporation the SST dipole is reinforced. This mechanism has been used to explain the maintenance of meridional modes of ocean–atmosphere covariability such as the Atlantic meridional mode (Moura and Shukla 1981; Chang et al. 1997) and the Pacific meridional mode (Chiang and Vimont 2004).

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Here, we show that interhemispheric antisymmetry is not necessary for meridional mode evolution and growth.

The mechanisms of growth of meridional modes have been the subject of several theoretical studies. Some of these studies have focused on modal (normal) growth of the least stable eigenvector of a dynamical system that contains the thermodynamic coupling (Noguchi 1998; Xie 1999; Xie et al. 1999; Wang 2010), while others have investigated the collective transient (nonnormal) growth due to positive interference between the individual modes (Vimont 2010; Martinez-Villalobos and Vimont 2016). The propagation characteristics of these kinds of modes have also been the subject of scholarly interest. Liu and Xie (1994) and Xie (1999) demonstrate that these coupled anomalies propagate westward and equatorward. The equatorward propagation occurs due to wind anomalies that are centered equatorward of SST anomalies, while the westward propagation occurs due to the westward phasing of the Rossby wave atmospheric response to the SST anomalies (Xie 1996). These propagation characteristics may have an important role in linking subtropical SST anomalies to the development of ENSO in the Pacific. As such, a theoretical understanding of the processes that grow and connect subtropical SST anomalies with the equatorial region may shed light on the nature of the observed connection between the Pacific meridional mode and ENSO evolution. As of now this connection is not completely understood, and is an area of active research (Vimont et al. 2003; Chang et al. 2007; Zhang et al. 2009; Larson and Kirtman 2013; Lin et al. 2015; Thomas and Vimont 2016).

Although the WES feedback is usually seen as responsible for equatorially antisymmetric variability, a few studies (Noguchi 1998; Xie 1999; Vimont 2010; Wang 2010; Martinez-Villalobos and Vimont 2016) have shown that the WES feedback is also able to maintain equatorially symmetric modes. More recently Clement et al. (2011), using an ensemble of general circulation models with “slab” ocean models (i.e., no Bjerknes feedback included), showed the existence in the equatorial Pacific of ENSO-like ocean–atmosphere modes maintained only by thermodynamic fluxes. It is possible that such a mode exists in nature, but its manifestation is overshadowed by the Bjerknes feedback.

This paper aims to provide basic insight into the similarities and differences between large-scale equatorially symmetric and antisymmetric (meridional mode–like) modes coupled by the WES feedback. Special attention will be given to analyzing mechanisms of propagation and growth for this two set of modes. While other studies have identified similar structures, propagation, and growth characteristics, this study emphasizes the role of transient processes in growth of meridional modes. As such, unlike the earlier referenced studies that focus on growth of a single eigenvector, we show that the interactions between modes (the transient processes) are critical for understanding meridional mode behavior. We note that we refer to these variations as “meridional modes” irrespective of their equatorial symmetry properties, and whether the structures are best described as a single mode or a transient process. While the name “meridional mode” is usually used to refer to variations that are equatorially antisymmetric (e.g., a meridional dipole), its use here in a more general sense is justified (as it will be argued later) because the growth and propagation of both equatorially symmetric and antisymmetric modes involve the same dynamics through the WES feedback. Further, the propagation in meridional wavenumber (which corresponds to a meridional propagation in physical space) further justifies the use of the term.

The remainder of the paper is organized as follows. Section 2 describes the model and methods used. Section 3 describes and discusses the results. Finally section 4 will present the conclusions.

2. Model description

The model used herein to investigate equatorially symmetric and antisymmetric variability consists of the atmospheric Gill–Matsuno model (Matsuno 1966; Gill 1980) coupled to a thermodynamic “slab” ocean. To simplify the calculations and facilitate the interpretation of the results we will adopt the “long-wave approximation” (Gill and Clarke 1974; Gill 1980) in the atmospheric equations, although in section 3e we will relax that assumption. For practical purposes the approximation consists of dropping the atmospheric damping in the v equation so the zonal wind is in geostrophic balance. In nondimensional form, this model may be written as follows:

\[ \begin{align*}
\epsilon u - yv &= -\frac{\partial \phi}{\partial x}, \\
yu &= -\frac{\partial \phi}{\partial y}, \\
\epsilon \phi + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= -K_q \, T, \\
\frac{\partial T}{\partial t} + \epsilon_r \, T &= au,
\end{align*} \]

where \( u \) and \( v \) denote low-level zonal and meridional wind anomalies, \( \phi \) denotes low-level geopotential...
anomalies, and $T$ denotes SST anomalies. In this model the atmosphere is diagnostic and completely determined by the underlying SST. The SST affects the atmosphere through deep heating, with the coupling parameter denoted as $K_q$. The atmosphere affects the SSTs through changes in evaporation due to zonal wind anomalies, with the parameter $\alpha$ controlling the strength of that coupling. The parameter $\alpha$ can be derived in a variety of ways: Czaja et al. (2002) and Vimont et al. (2009) derive $\alpha$ using a Taylor expansion of the bulk latent heat flux formula around variations in the zonal wind $u$. Notice that in the presence of mean easterlies ($\alpha > 0$), a positive $u$ anomaly implies a reduction in wind speed and hence evaporation, and thus induces a positive SST tendency. We take both the $K_q$ and $\alpha$ parameters to be constant, although the model could be easily expanded to include a spatial dependence of $K_q$ or $\alpha$ (this is discussed further in section 4b). Both SST and atmospheric parameters are linearly damped by terms $\varepsilon_T$ and $\varepsilon$, respectively. The effective coupling of the system is given by $K_q \alpha$ (Martinez-Villalobos and Vimont 2016). The parameter values are shown in Table 1. As we are interested in large-scale variability, the Gill–Matsuno atmospheric model is appropriate, although Vimont (2010) investigates a similar model framework using both the Gill–Matsuno (no-long-wave approximation used) and Battisti et al. (1999) reduced-gravity boundary layer models. Remaining details of the model are described in Vimont (2010) and Martinez-Villalobos and Vimont (2016).

Note that in Eq. (1) the SST tendency is coupled to the atmosphere only through zonal wind anomalies $u$, and that the steady atmospheric response can in turn be expressed entirely in terms of temperature. We express the atmosphere in terms of the SST alone by decomposing the SST in the meridional direction using parabolic cylinder functions $\psi_m(y)$ (see appendix A):

$$T(t, y, x) = \sum_{m=0}^{\infty} T_m(t) \psi_m(y) \exp(ikx),$$

where $k$ is the zonal wavenumber considered. A particular parabolic cylinder function $\psi_m$ has $m$ zeros, has the symmetry of $m$ ($m$ even corresponds to equatorially symmetric structures, and $m$ odd to equatorially antisymmetric structures), and has loadings increasingly farther from the equator as $m$ increases (see Fig. A1). So, a low-latitude SST signal is dominated by low $m$ modes, while a high-latitude signal is increasingly dominated by high $m$ modes.

The atmosphere may be written in terms of the SSTs (see appendix B) as

$$u(t, y, x) = \frac{1}{2} \left( q_0(t) \phi_0(y) + \sum_{m=1}^{\infty} q_{m+1}(t) \left[ \psi_{m+1}(y) - \sqrt{\frac{m+1}{m}} \psi_{m-1}(y) \right] \right) \exp(ikx),$$

$$v(t, y, x) = \sum_{m=0}^{\infty} \frac{1}{2(2m+1)} K_q T_{m+1}(t) \left[ \varepsilon + ik \right] \psi_m(y) \exp(ikx),$$

$$\phi(t, y, x) = \frac{1}{2} \left( q_0(t) \phi_0(y) + \sum_{m=1}^{\infty} q_{m+1}(t) \left[ \psi_{m+1}(y) + \sqrt{\frac{m+1}{m}} \psi_{m-1}(y) \right] \right) \exp(ikx),$$

where $q_0$ and $q_{m+1}$ are the amplitudes of the atmospheric Kelvin wave and $m$th Rossby wave respectively. Here, we use the terms “Kelvin wave” and “Rossby wave” in the same manner as Gill (1980): to describe features of the steady zonal and meridional atmospheric response to a specific heating structure (in our case, $K_q T$). The amplitudes of these waves are determined by the SST as

$$q_0 = -K_q \frac{T_0}{\varepsilon + ik},$$

$$q_{m+1} = K_q \left[ \sqrt{\frac{m(m+1)}{m+1}} T_{m+1} + \frac{mT_m}{-(2m+1)\varepsilon + ik} \right].$$

Note that the denominator in Eq. (4) includes a term related to the inverse of the nondimensional group velocity.
for nondispersive (due to the long-wave approximation) Kelvin (+1) or Rossby [−(2m + 1)] waves (Gill 1980). Again, as in Gill (1980), the group velocity determines the meridional extent of the response, and in our case the zonal phase difference between the heating (i.e., SST) and the zonal wind component of the atmospheric response (which determines the sign of the WES feedback, and depends on the angle of the complex denominator). These terms will prove useful in tracking the role of Rossby or Kelvin waves in the system evolution.

Using Eqs. (3) and (4) in the SST equation for Eq. (1) we may write an equation for a particular \( T_m \) just in terms of SST modes:

\[
\frac{\partial T_m}{\partial t} = \frac{1}{2} K_a \alpha \left\{ \frac{\sqrt{(m-1)m}}{-(2m-1)e + ik} T_{m-2} + \left[ \frac{m-1}{-(2m-1)e + ik} - \frac{m+2}{-(2m+3)e + ik} \right] T_m - \sqrt{(m+1)(m+2)} \frac{T_{m+2}}{-(2m+3)e + ik} \right\} - e_T T_m. \tag{5}
\]

Also an equation for the squared amplitude of a particular mode can be derived [see Eq. (C3)]. We defer the analysis of the terms in these equations to section 3b. For now, it is important to notice that the effect of the WES feedback in the evolution of SST anomalies is enclosed in the \( K_a \alpha \) term, and the SST damping is just the last term. Also, Eqs. (4) and (5) show explicitly that \( T_m \) is affected by Rossby wave \( m - 1 \) (or Kelvin wave for \( T_0 \)) and \( m + 1 \) (note the \( 2m - 1 \) or \( 2m + 3 \) terms in the denominator, which relate to the Rossby wave group velocity as discussed above), and that symmetric and antisymmetric modes do not interact. This is a consequence of \( K_a \) and \( \alpha \) being symmetric. Vimont (2010) and Martinez-Villalobos and Vimont (2016) have studied how meridional variations in the coupling affect thermodynamically coupled variability.

Because symmetric and antisymmetric modes do not interact, we can analyze them separately. Figure 1 shows an example of equatorially symmetric and antisymmetric structures and how they evolve in time. Also plotted is their meridional extent of the SST structures and how they evolve in time. Also plotted is their interaction, we can analyze them separately.

### 3. Results and discussion

#### a. Low and high SST modes

In this section we will analyze the evolution of a particular \( T_m \) mode from Eq. (5) for low and high \( m \) SST modes. We show that a particular mode will propagate “outward” to higher- and lower-order modes, while retaining symmetry. This continues until the lowest-order mode, which excites either the Kelvin wave (symmetric) or mixed Rossby–gravity wave (antisymmetric), with fundamental consequences on the growth rate of symmetric or antisymmetric structures.

From Eq. (5) we notice that for \( m \geq 2 \) both symmetric \((m\ even)\) and antisymmetric \((m\ odd)\) SST modes evolve in a qualitatively similar way, via excitation of the \( m + 1 \) and \( m - 1 \) Rossby waves. Note that the \( T_{m+2} \) structure excites the \( m + 1 \) and \( m + 3 \) Rossby waves [see Eq. (4)]; the former includes \( u \) anomalies that project onto \( \psi_m \) [last term multiplying \( K_q \alpha \) in Eq. (5)]. Similarly, the \( T_{m-2} \) structure excites the \( m - 3 \) and \( m - 1 \) Rossby waves [see Eq. (4)]; the latter includes \( u \) anomalies that project onto \( \psi_m \) [middle term multiplying \( K_q \alpha \) in Eq. (5)]. And finally, the \( T_m \) structure itself excites both \( m - 1 \) and \( m + 1 \) Rossby waves, which both include \( u \) anomalies that project onto \( \psi_m \) [first term multiplying \( K_q \alpha \) in Eq. (5)]. This demonstrates that for \( m \geq 2 \) a particular symmetric or antisymmetric mode will propagate “outward” (toward the next higher and next lower symmetric or antisymmetric mode, respectively) through Rossby wave excitation. In physical space, the propagation toward different modes represents a meridional propagation toward a higher-latitude variation (for \( m \) increasing) or toward more equatorially confined variability (for \( m \) decreasing).

For \( m = 0 \) and \( m = 1 \) the situation is different. In this case the evolution of both \( T_0 \) and \( T_1 \) modes are dictated by qualitatively different equations:

\[
\frac{\partial T_0}{\partial t} = \frac{1}{2} K_a \alpha \left[ \frac{-1}{e + ik} - \frac{2}{-3e + ik} \right] T_0 - \frac{\sqrt{2}}{-3e + ik} T_2 - e_T T_0, \tag{6a}
\]

\[
\frac{\partial T_1}{\partial t} = \frac{1}{2} K_a \alpha \left[ \frac{-3}{-5e + ik} T_1 - \frac{\sqrt{6}}{-5e + ik} T_3 \right] - e_T T_1. \tag{6b}
\]

The evolution of the \( T_0 \) term is affected by both atmospheric Kelvin wave and the first symmetric Rossby
wave, whereas the $T_1$ term is affected by just the first antisymmetric Rossby wave [the next-lower antisymmetric mode is the mixed Rossby–gravity wave, for which $u$ anomalies under the long-wave approximation are strictly zero as shown in Eq. (4) (i.e., $q_1 = 0$)].

Taken together, the full evolution from $m = 2$ toward $m = 0, 1$ modes illustrates two important characteristics of thermodynamically coupled variability. First, for $m = 2$ the dynamics of both equatorially symmetric and antisymmetric SST patterns is essentially the same. Second, the dynamics of the system only differ for the lowest-order modes $m = 0, 1$, where the Kelvin (for $m = 0$) and mixed Rossby–gravity (for $m = 1$) modes have fundamentally different effects. Note from Fig. 1 that the total growth for the symmetric and antisymmetric patterns is quite different. We will demonstrate below that the reason for this difference lies in the atmospheric Kelvin wave.

b. Analysis of terms in SST equation

To analyze Eq. (5) in the parameter space considered herein we rewrite it in terms of the ratio between the wavenumber and atmospheric damping $\nu$, and the ratio between the coupling and damping terms ($\sigma$ the stability parameter):

$$\nu = \frac{k}{\varepsilon} \quad \text{and} \quad \sigma = \frac{K}{\varepsilon} \frac{\alpha}{\varepsilon \sigma}.$$  (7)
as

\[
\frac{\partial T_m}{\partial t} = \frac{1}{2} \varepsilon_r \left[ -h(m-1, \nu) \sigma T_{m-2} + f(m, \sigma, \nu) T_m + h(m+1, \nu) \sigma T_{m+2} \right].
\]

where the functions introduced are defined as

\[
h(m, \nu) = \frac{\sqrt{m(m+1)}}{(2m+1) - i\nu},
\]

\[
f(m, \sigma, \nu) = g(m, \nu) \sigma - 2, \quad \text{and}
\]

\[
g(m, \nu) = \frac{(2m+1) - 3i\nu}{(2m-1)(2m+3) - 2i\nu(2m+1) - \nu^2}.
\]

The function \(h(m, \nu)\), the exchange function, encodes part of the influence of Rossby wave \(m\) [the atmospheric Kelvin wave does not participate in the exchange as \(h(-1, \nu) = 0\)] via connecting \(T_{m-2}\) and \(T_{m+2}\) to \(T_m\), and its real part is always positive. Notice that the sign of \(h(m, \nu)\) in Eq. (8) shows that both \(T_{m-2}\) and \(T_{m+2}\) terms affect \(T_m\) evolution in different directions (opposite polarity). This will prove important in explaining the propagation characteristics and transient growth of the modes.

Function \(f(m, \sigma, \nu)\) contains the effect of \(T_m\) on itself. Here, the function \(g(m, \nu)\) contains part of the WES feedback influence on growth (if \(g > 0\) it contributes to \(T_m\) growth, and if \(g < 0\) it contributes to \(T_m\) decay) and the \(-2\) term corresponds to the SST damping. Notice that the instantaneous growth or decay of a single \(T_m\) mode depends on the balance between the WES feedback \([g(m, \nu)\sigma]\) and the SST damping \((-2)\), where here we distinguish the WES feedback as including only the wind-induced change in surface latent heat flux.

c. Long zonal wavelength limit

As a starting point we will study the long zonal wavelength \((k \to 0)\) limit (i.e., \(\nu = 0\)). This limit will prove useful in understanding the more general variability for long but finite zonal wavelengths. In this case Eq. (5) [also Eq. (8)] is reduced to

\[
\frac{\partial T_m}{\partial t} = \frac{1}{2} \varepsilon_r \left[ -h(m-1, 0) \sigma T_{m-2} + f(m, \sigma, 0) T_m + h(m+1, 0) \sigma T_{m+2} \right].
\]

Functions \(h\) and \(f\) in this limit are purely real, as in this limit atmospheric waves are zonally in phase with the SST:

\[
h(m, 0) = \frac{\sqrt{m(m+1)}}{(2m+1)} \quad \text{and}
\]

\[
f(m, \sigma, 0) = g(m, 0) \sigma - 2 = \frac{(2m+1)\sigma}{(2m-1)(2m+3)} - 2.
\]

In this case, the equation for a particular mode growth [see Eq. (C3)] is vastly simplified to

\[
\frac{\partial |T_m|^2}{\partial t} = \varepsilon_r \left[ -h(m-1, 0) \sigma T_{m-2} T_m + f(m, \sigma, 0) |T_m|^2 \right] + h(m+1, 0) \sigma T_{m+2} T_m.
\]

Figure 2a shows \(h(m, 0)\) as a function of mode \(m\). This shows that \(h\) is a very flat function of \(m\) for \(m > 1\) \(h(m - 1, 0) = 0\) for \(m = 0, 1\). The difference in value between \(h(m + 1, 0)\) and \(h(m - 1, 0)\) is positive and small, and gets progressively smaller as \(m\) increases \(|dh(m, 0)/dm| \sim 1/16m^2 > 0\). This implies that if \(T_{m-2}\) has the same magnitude as \(T_{m+2}\), then their effect on \(T_m\) growth roughly cancels out, although the effect...
of $T_{m+2}$ slightly dominates. This dominant influence of the mode $m+2$ on the lower-order $m$ helps explain the equatorward propagation (i.e., propagation toward lower $m$) seen in Fig. 1.

Next, consider the growth equations for modes $m-2$ and $m+2$:

$$\frac{\partial |T_{m-2}|^2}{\partial t} = e_T [-h(m-3,0)\sigma T_{m-4} T_{m-2} + f(m-2,\sigma,0) |T_{m-2}|^2 + h(m-1,0)\sigma T_{m-2}] \quad \text{and} \quad (13)$$

$$\frac{\partial |T_{m+2}|^2}{\partial t} = e_T [-h(m+1,0)\sigma T_{m} T_{m+2} + f(m+2,\sigma,0) |T_{m+2}|^2 + h(m+3,0)\sigma T_{m+4} T_{m+2}] \quad \text{(14)}$$

We notice that the third term in Eq. (13) is exactly equal to the first term in Eq. (12), but with opposite sign; similarly with the first term of Eq. (14) and third term in Eq. (12). Without loss of generality also consider that $T_{m-2}$, $T_m$, and $T_{m+2}$ all start positive. This shows that a positive $T_{m+2}$ will grow $T_m$ amplitude, while at the same time $T_m$ will decrease $T_{m+2}$ amplitude. Then $T_m$ will increase $T_{m-2}$ amplitude while $T_{m-2}$ will damp $T_m$. As a consequence of this process $T_{m+2}$ grows $T_m$ that then grows $T_{m-2}$. This, at the core, is the mechanism that explains equatorward propagation (propagation toward lower-order modes) of thermodynamically coupled variability.

The $f(m,\sigma,0)$ function in Eqs. (10) and (11) is a key in the prospect of long-term growth of a particular mode. This function depends on the particular mode $m$ and the ratio between coupling and damping, the stability parameter $\sigma$. A particular mode $m$ will grow (at least for some time) if $f(m,\sigma,0)>0$; otherwise, it decays. Figure 2b shows how $f(m,\sigma,0)$ varies with $m$ (for the first 10 modes), for different values of the stability parameter. Here $\sigma_0 = 4.83$ using the standard values shown in Table 1. For the $m$ values of interest $g(m,0)$ has a pole at $m = \frac{1}{2}$ implying, as shown in Fig. 2 for three different values of $\sigma$, a very different behavior for modes $m=0$ and $m=1$. In this limit $g(m,0)$ peaks at $m=1$, is positive for $m \geq 1$, and is negative just when $m = 0$. This shows that the WES feedback is positive (it works toward amplifying an initial $T_m$ anomaly) for all $m$ modes except $m=0$. The function $g(m,0)$ gets monotonically smaller as $m$ increases, implying that the WES feedback gets progressively less effective as we move to higher latitudes, consistent with Vimont (2010).

For $m \geq 1$ there is a tension in function $f(m,\sigma,0)$ between a positive WES feedback represented by the $g(m,0)\sigma$ term, that acts to enhance $|T_m|^2$ growth, and the SST damping represented by the $-2$ term, that acts to damp SST. When $g(m,0)\sigma$ is greater than 2 there is a prospect for transient growth. In the cases shown in Fig. 2b, $f(m,0.5\sigma_0,0) < 0$ for all values of $m$ so the system simply decays, although at a rate slower than $e_T^{-1}$. For $\sigma = \sigma_0, f(m,\sigma,0) > 0$ just for $m = 1$ so the system will grow, at least for some time, just when the SSTS project onto $\psi_1$. Finally, for $\sigma = 2\sigma_0, f(m,2\sigma_0,0) > 0$ for $m = 1,2$, so the SSTS will grow if they project onto $\psi_1$ and $\psi_2$, but still the greatest growth will be achieved when they project onto $\psi_1$. For $m = 0, g(m,0)\sigma < 0$, so the WES feedback turns into a negative feedback, and the system is additionally damped for that mode.

The reason why $T_0$ is damped by, and $T_1$ grows through, the WES feedback lies in the differences between atmospheric Kelvin and Rossby waves. Consider an equatorially symmetric zonally homogeneous positive SST anomaly confined close to the equator. The atmospheric response includes [see Eq. (4) for $\bar{q}_0$ and $\bar{q}_2$] a positive zonal wind anomaly $u_R$ associated with the first symmetric Rossby wave on top of the anomaly (recall that there is zero zonal phase difference between SST and the atmosphere in the $\nu = 0$ case) and a negative zonal wind anomaly $u_K$ associated with the Kelvin wave. Since the mean zonal winds are negative along the equator ($\alpha < 0$), the Rossby response acts to decrease evaporation, further increasing the positive SST anomaly, whereas the Kelvin wave acts to increase evaporation, damping the SST anomaly.

This result implies a very different situation between symmetric and antisymmetric SST patterns with respect to growth. A subtropical antisymmetric pattern will propagate equatorward (toward lower $m$) when it will finally project onto $\psi_1$. At that point the system will grow while at the same time causing a $T_3$ tendency in the opposite direction [see Eq. (12) for $m = 1$]. As $T_3$ grows in the opposite direction, it ultimately counteracts the growth of $T_1$, causing the entire system to decay back to zero. On the other hand, a subtropical equatorially symmetric pattern will also propagate toward lower $m$, finally projecting onto $\bar{q}_0$ where it will be additionally damped by the WES feedback due to the atmospheric Kelvin wave effect [see Eq. (6) with $k = 0$].

To illustrate how the Kelvin wave affects growth, we will consider the $\sigma = \sigma_0$ case from here on. In this case just the $m = 1$ mode has prospective growth. For the rest of the paper we will use the first 10 $T_m$ modes in our calculations (five symmetric and five antisymmetric). We have tested that results do not change qualitatively by increasing the number of modes retained. Figure 3 shows how this asymmetry in the evolution of equatorially symmetric and antisymmetric SST anomalies in
this limit plays out. To make things easier to interpret, the system will be initialized as $T(0, y) = c(y)$ in the symmetric case and $T(0, y) = \psi_d(y)$ for the antisymmetric case. In accordance with (10) both patterns start very similarly. The symmetric (antisymmetric) pattern generates a positive $T_2$ ($T_3$) response, and a similar but smaller negative $T_6$ ($T_7$) response, while being damped by these modes in the process. Both positive signals continue propagating equatorward, losing energy in the process until the antisymmetric SST pattern projects onto $\psi_1$, where the system grows for some time, and the symmetric SST pattern project onto $\psi_0$, where the system decays even more rapidly. Notice how similar Figs. 3a and 3b are, except in the $T_0$ and $T_1$ evolution.

This general pattern is confirmed in Fig. 4. This figure shows how the symmetric initial structure $T(0, y) = \psi_d(y)$ and antisymmetric initial structure $T(0, y) = \psi_5(y)$ evolve spatially in the meridional direction from 0 to 300 days. Also shown in this figure is how the symmetric pattern would evolve if we artificially suppress the atmospheric Kelvin wave in the SST calculation. This suppression consists of neglecting the $-1/(\varepsilon + ik)$ ($k = 0$) term in the $T_0$ evolution calculation [see Eq. (6)].

**Fig. 3.** Evolution of individual $T_m$ coefficients for the (a) symmetric initial condition $T(0) = \psi_d(y)$ and (b) antisymmetric initial condition $T(0) = \psi_5(y)$. Both panels are shown for stability parameter $\sigma_0$ and for $k = 0$.

**Fig. 4.** Latitude–time evolution of SST from 0 to 300 days. Evolution of the (a) antisymmetric initial condition $T(0) = \psi_5(y)$ and (b) symmetric initial condition $T(0) = \psi_d(y)$. (c) As in (b), but the atmospheric Kelvin wave is suppressed. Shading represents SST. In this $k = 0$ case the evolution does not depend on longitude.
toward the equator from day 0 to about day 150 (notice the decrease in the shading) until the SSTs project onto $\psi_1(y)$ close to the equator where growth increases again (notice the shading starting at day 150). In contrast, Fig. 4b shows the amplitude of the symmetric pattern simply decaying with time. We compare this pattern with Fig. 4c, which shows how the symmetric structure evolves without the atmospheric Kelvin wave. The damping effect of the Kelvin wave is apparent from about day 100: while the symmetric structure equatorially grows again (notice the shading starting at day 150). In contrast, Fig. 4b shows the amplitude of the symmetric pattern simply decaying with time. We compare this pattern with Fig. 4c, which shows how the symmetric structure evolves without the atmospheric Kelvin wave. The damping effect of the Kelvin wave is apparent from about day 100: while the symmetric structure equatorially grows again (notice the shading starting at day 150).

Figure 5 shows the total SST growth evolution for (a) the antisymmetric initial condition $T(0) = \psi_1(y)$, (b) the symmetric initial condition $T(0) = \psi_2(y)$, and (c) $T(0) = \psi_4(y)$ [as in (b)] except that the atmospheric Kelvin wave is suppressed. All panels are shown for stability parameter $\sigma_0$ and for $k = 0$.

1) TOTAL SST GROWTH

From the SST equation [Eq. (1)] we calculate an equation for the growth of total SST anomalies. In this $\nu = 0$ limit [see Eq. (C6) for the general growth equation], and considering Eq. (12), this is equal to

$$\frac{\partial(T^2)}{\partial t} = L_x \sum_{m=0}^{\infty} \frac{\partial|T_m|^2}{\partial t} = L_x \varepsilon_{T} \sum_{m=0}^{\infty} f(m, \sigma, 0)|T_m|^2,$$

(15)

where the angle brackets $\langle \cdot \rangle$ mean integration over the basin considered, and $L_x$ is its zonal extension. The exchange terms in Eq. (12) $\langle h \rangle$ get canceled out when we sum all the $\partial T_m^2/\partial t$ terms. The total SST growth evolution is described by this equation, together with Eq. (12) that determines the growth of an individual mode. Equations (15) and (12) indicate that in the $\nu = 0$ limit the role of the exchange function $h(m, 0)$ is to propagate the signal toward lower modes (curbing growth in the process, as explained in next section), and the growth function $f(m, \sigma, 0)$ determines whether that signal transiently grows or decays.

Figure 6 shows the total growth $\langle T^2 \rangle$ for antisymmetric initial conditions $T(0) = \psi_1(y)$, $\psi_3(y)$, $\psi_5(y)$, $\psi_7(y)$, $\psi_9(y)$ (Fig. 6a), symmetric initial conditions $T(0) = \psi_0(y)$, $\psi_2(y)$, $\psi_4(y)$, $\psi_6(y)$, $\psi_8(y)$ (Fig. 6b), and the same symmetric initial conditions with the atmospheric Kelvin wave suppressed (Fig. 6c). We notice
that the $T_1$ antisymmetric initial condition grows for some time (about 100 days) before it is damped by $T_3$ [see Eq. (12)]. All the other antisymmetric initial conditions decay until the signal propagates close enough to the equator that it projects significantly onto $\psi_1(y)$, at which point the system experiences some growth, leading to secondary peaks that are less energetic and peak later in time (e.g., around 190 days for $T_3$ and 250 days for $T_5$). The symmetric initial conditions all decay, with $T_0$ decaying the fastest (due to the atmospheric Kelvin wave damping effect) and $T_2$ the slowest. This is consistent with the values of $f(m, \sigma_0, 0)$ shown in Fig. 2b.

2) NORMAL VERSUS NONNORMAL GROWTH

In the limit $\nu \to 0$ we may write Eq. (10) in matrix form

$$\frac{\partial T}{\partial t} = MT = -i\omega T,$$

(16)

where $T = (T_0, T_1, T_2, \ldots) \exp(-i\omega t)$, and $M$ entries are given by

$$M_{m,m-2} = \frac{1}{2} e^{-i} h(m-1,0)\sigma,$$

$$M_{m,m} = \frac{1}{2} f(m,\sigma,0),$$

$$M_{m,m+2} = \frac{1}{2} e^{i} h(m+1,0)\sigma.$$  

(17)

As previously stated we truncate at $m = m_T = 9$ in our calculations, so $M$ is a $10 \times 10$ matrix. The term $M_{0,11}$ in Eq. (17) is neglected. It has been corroborated numerically (not shown, but similar to the explanation in section 3e) that this choice does not affect the results. For $\sigma = \sigma_0$ the spectrum of $M$ is such that the system is linearly stable [i.e., $\text{Im}(\omega) < 0$ for all $\omega$] and nonnormal (i.e., $M^T M \neq MM^T$), so growth may be achieved transiently (Farrell and Ioannou 1996). This also implies that $\lim_{t\to\infty} T = 0$ as is the case in previous plots.

The $\nu \to 0$ limit provides a convenient way to show the differences between normal and nonnormal growth. The system is nonnormal due to the exchange function $h$ being nonzero and of opposite sign for $M_{m+2,m}$ and $M_{m,m+2}$ [Eq. (17)]. Artificially suppressing nonnormality by zeroing out the exchange functions, the equation that determines the evolution of a particular mode amplitude is now equal to [cf. Eq. (10)]:

![Fig. 6. Total SST growth for (a) antisymmetric initial conditions, (b) symmetric initial conditions, and (c) symmetric initial conditions [as in (b)] except that the atmospheric Kelvin wave is suppressed. For (a) the lines show the total growth for initial conditions starting at $T_1$ (solid), $T_3$ (dashed), $T_5$ (dotted), $T_7$ (dashed–dotted), and $T_9$ (plus sign). For (b) and (c) the lines show the total growth for initial conditions starting at $T_0$ (solid), $T_2$ (dashed), $T_4$ (dotted), $T_6$ (dashed–dotted), and $T_8$ (plus sign). All panels are shown for stability parameter $\sigma_0$ and for $k = 0$.](image-url)
The equation for total SST growth [Eq. (15)] remains unchanged, and actually contains the same information as Eq. (18). Note, however, that in the normal system (no exchange terms) each mode can be integrated individually using Eq. (18) and the result summed to obtain total growth [the integration of Eq. (15)]. The solution to Eq. (18) is

\[
T_m(t) = T_m(0) \exp\left[\frac{1}{2} \varepsilon_T f(m, \sigma, 0)t\right].
\]  

(19)

We observe that without the exchange terms the system will just decay or grow indefinitely depending on the sign of \(f\). That is, for \(\sigma = \sigma_0\) the system will grow indefinitely if the initial SSTs project onto \(\psi_1(y)\); otherwise, it will decay. In reality, the WES feedback coupling is “nonnormal” so the growth is curbed by the exchange terms. In that case the SSTs will grow for some time if they project onto \(\psi_1\), but the growth will be limited by the effect of the exchange terms.

Figure 7 shows the total growth for both normal [Eq. (18)] and nonnormal [Eq. (10)] cases for the antisymmetric initial condition \(T(0) = \psi_1(y)\) and the symmetric initial condition \(T(0) = \psi_0(y)\). In the normal case the antisymmetric initial condition grows exponentially, and the symmetric initial condition decays exponentially. In the nonnormal case the growth of the antisymmetric initial condition initially exceeds growth of the normal system due to growth of \(T_3\) and higher modes, but is eventually curbed by the interaction with \(T_3\) [the exchange term \(\varepsilon_T h(2, 0)\sigma_0 T_1 T_3\) in Eq. (10)]. Similar arguments hold for the symmetric case \(T_0\), which is additionally damped by \(T_2\) and the system decays even more rapidly than the normal case. Thus, nonnormality contributes to both short-term growth in the system, as well as eventual decay.

d. Finite large-scale zonal variability

The \(k = 0\) limit is useful in understanding the low-frequency large-scale thermodynamically coupled variability. This section will describe how long but finite large-scale variability deviates from this idealized case. As was done for the \(k = 0\) case [Eq. (16)] we can study the stability of the system by constructing a dynamical matrix that contains the WES feedback coupling [Eq. (1)] for a general \(k\) value and analyze its eigenvalues. The matrix is similar to Eq. (17), but we use the general functions shown in Eq. (9). That is,

\[
M_{m,m-2} = -\frac{1}{2} \varepsilon_T h(m-1, \nu)\sigma,
\]

\[
M_{m,m} = \frac{1}{2} f(m, \sigma, \nu),
\]

\[
M_{m,m+2} = \frac{1}{2} \varepsilon_T h(m+1, \nu)\sigma.
\]  

(20)

Note that here we retain the nonnormality inherent to the system. Solving Eq. (16) for this dynamical matrix, we find that an initial condition \(T(0) = [T_0(0), T_1(0), T_2(0), \ldots]\) evolves as

\[
T(t) = \exp(\mathbf{M}t) T(0) = \mathbf{G}(t) T(0).
\]

(21)

where \(\mathbf{G}(t) = \exp(\mathbf{M}t)\) is the Green’s function. This shows that if all \(\mathbf{M}\) eigenvalues \(\text{Im}(\omega)\) are negative [see Eq. (16)] the system is linearly stable and asymptotically decays (although the initial SST anomalies may grow for a finite time due to nonnormality as shown in
the previous section for \( k = 0 \). If at least one \( \omega \) is positive, then the system is linearly unstable, and will be dominated by exponentially growing modes (akin to the normal case shown in Fig. 7).

Figure 8 shows \( \text{Im}(\omega) \) for the least stable eigenvector of the system as a function of \( \nu = k/e \) for equatorially antisymmetric case, symmetric case, and symmetric with the atmospheric Kelvin wave suppressed case. For context a zonal wavenumber \( k = 2\pi/120^\circ \) and \( e = (2 \text{day})^{-1} \) corresponds to \( \nu = 2.44 \). For the standard parameters shown in Table 1 the system is linearly stable for large zonal-scale variability. The system does become unstable for larger \( k \) (and stable again for even bigger \( k \)) but that is well outside the validity of the Gill–Matsuno model under the long-wave approximation. The system is stable at all \( k \) when the long-wave approximation is not used (not shown; see section 3e for a description of the model used for that effect).

We observe that the stability of the system is greatly reduced for equatorially symmetric variability when we suppress the atmospheric Kelvin wave for very long zonal wavelengths up to \( \nu \sim 3.9 \). After that point the absence of the Kelvin wave does not seem to affect the stability of the system: note how the dashed and dash-dotted curves converge in Fig. 8b. The atmospheric Kelvin wave gets progressively less important in the dynamics of the system as \( \nu \) increases. Figure 9 shows that the amplitudes of the real part of the growth function \( f(m = 0, \sigma_0, \nu) \) with and without the Kelvin wave get closer for increasing \( \nu \). Consequently the destructive effect on growth due to the Kelvin wave gets less important as we deviate more and more from the \( k = 0 \) idealized case. This is simply a result of phasing of the Kelvin wave response with respect of the initial forcing: as \( k \) gets larger or \( e \) smaller, the Kelvin wave response becomes out of phase with the forcing [see Eq. (6)]. Both with and without the atmospheric Kelvin wave \( f(m = 0, \sigma_0, \nu) \), values should asymptote to \(-2 \) [see Eq. (9)] as \( \nu \to 0 \).

Overall, the picture gets much more complicated for nonzero zonal wavenumber. Now the evolution of a particular mode \( T_m \) will depend on the real and imaginary part of the growth function \( f(m, \sigma, \nu) \) as well as in the phasing in the complex plane with modes \( T_{m-2} \) and \( T_{m+2} \). Nonetheless, the general lessons learned in the \( k = 0 \) case remain valid (albeit much less pronounced) for large-scale dynamics.

Figure 10 shows the maximum transient growth for modes that are equatorially antisymmetric, symmetric, and symmetric with atmospheric Kelvin wave suppressed as a function of \( \nu \). Only the linearly stable regime is considered. The framework used to calculate the optimal initial conditions that maximize transient growth in each case is described in appendix D (see also Farrell and Ioannou 1996; Vimont 2010; Martinez-Villalobos and Vimont 2016). We observe that the maximum growth for antisymmetric and symmetric variability starts converging as \( \nu \) increases. The damping provided by the atmospheric Kelvin wave is still important, but relatively less so as the zonal wavelength gets shorter. This effect is emphasized by comparing the maximum growth with and without the Kelvin wave influence in Fig. 10.

The reason for the decrease of damping provided by the Kelvin wave as \( \nu \) increases, and consequently the increase in similarity between symmetric and
antisymmetric structures growth due to the WES feedback, is explained by the relative phasing between the SST and atmospheric response for a finite \( \nu \). Figure 11 illustrates how the first Rossby wave contributes to growth, and the Kelvin wave contributes to damping for a zonal wavelength of 120° (\( \nu = 2.44 \)). Notice in Fig. 10 that for this \( \nu \) antisymmetric structures still grow more than symmetric ones, but the gap has decreased tremendously compared to the \( k = 0 \) case. Consequently, we expect a diminished (compared to the \( k = 0 \) case), but still sizable contribution to damping on average by the Kelvin wave.

Figure 11a shows the instantaneous wind anomaly response to a sea surface temperature anomaly pattern of the form \( T(x, y) = \psi_0(y) \exp(ikx) \) associated with the Kelvin wave. Focusing first on the positive SST anomaly located from \( x = 0.25 \) to 0.75 (\( x \) in \( 2\pi/k \) units) in the zonal axis, we notice that over the western part of this pattern (from \( x = 0.25 \) to 0.45) the total wind relaxes (a positive wind anomaly implies a relaxation of the easterly trades), while in the eastern part (from \( x = 0.45 \) to 0.75) the total wind magnitude increases. The relaxation of the wind in the western part implies a decrease in evaporation and a tendency for the SST anomaly to grow there, while in the eastern part the reinforcement of the trades implies a cooling of the warm anomaly. In other words, there is SST growth in the western part and damping in the eastern part. The Kelvin wave acts to damp the anomaly on average

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**Fig. 10.** (a) Maximum growth \( \mu \) as a function of \( \nu \). (b) Time when maximum growth is achieved as a function of \( \nu \). The plots cover the linearly stable range under the long-wave approximation.

**Fig. 11.** Instantaneous atmospheric wind (vectors) and low-level geopotential (contours) response to \( \text{SST} = \psi_0(y) \exp(ikx) \). Shown are the (a) Kelvin wave, (b) first Rossby wave, and (c) total instantaneous atmospheric response. Plots correspond to a zonal wavenumber \( L_x = 120° \) corresponding to \( \nu = 2.44 \), and \( L_y = 10° \) (from parameters in Table 1). SST is shaded, and the solid (dashed) contours correspond to positive (negative) low level geopotential anomalies. Units are arbitrary, but consistent between panels.
because for this \( \nu \) value the phasing is such that the negative wind anomaly associated with the Kelvin wave overlaps more of the warm anomaly (the eastern part is bigger in extension than the western part). A similar reasoning explains that on average the first Rossby wave acts to grow the anomaly. In this case, the reduced group velocity of the atmospheric Rossby wave means that the wind relaxes over most of the warm SST anomaly (Fig. 11b). Adding the responses (Fig. 11c) shows that in this regime the WES feedback acts to grow the initial anomaly. Notice also that this mechanism implies a westward propagation of the structure.

In the \( \nu = 0 \) limit the Kelvin and Rossby wave responses are exactly zonally in phase with the SST anomalies. In that limit, both negative wind anomalies associated with the Kelvin wave, and positive wind anomalies associated with the first Rossby wave perfectly overlap a SST warm anomaly in the zonal direction, and SST anomalies grow or decay in place with no zonal propagation. For a finite zonal wavelength the phasing of the Kelvin wave changes more rapidly as \( \nu \) increases than the phasing induced by the atmospheric Rossby wave, due to the differing group velocities. As a consequence, the Kelvin damping gets reduced more rapidly than the Rossby growth, so on average the symmetric SST mode grows more for shorter, but still long, zonal scales. In summary, in the zonal long-wave range, as \( \nu \) increases from 0 the equatorially symmetric mode propagates more efficiently westward achieving more growth in the process compared to smaller \( \nu \).

The damping produced by the Kelvin wave is certainly smaller for finite \( \nu = k/a \), but it is still sizable. Figure 12b shows the evolution of the symmetric structure shown in Fig. 1a with the atmospheric Kelvin wave suppressed. The growth in this case is vastly larger (Fig. 12c; also cf. Figs. 12b and 1c) showing the importance of this mechanism in damping symmetric structures. This explains why equatorially antisymmetric large-scale patterns are preferentially excited by the WES feedback: for symmetric structures, the Kelvin wave plays a critical role in damping variability, while there is no analogous mechanism for anti-symmetric structures.

**e. The use of the long-wave approximation**

In any analytical approach some allowance is needed in order to find the adequate balance between analytical understanding and realism. In this study we take an analytical look on the equations that govern thermodynamically coupled variability in hopes of gaining insight on the differences and similarities between equatorially symmetric and antisymmetric modes. To simplify the mathematics and interpretation of the variability we use the long-wave approximation. This approximation should give us a qualitative understanding of the variability in the vicinity of the \( k = 0 \) region. Here, we test the validity of the major conclusions.
away from the $k = 0$ limit by relaxing the long-wave approximation.

Vimont (2010) and Martinez-Villalobos and Vimont (2016) numerically analyze a Gill–Matsuno atmospheric model coupled to a slab ocean without using the long-wave approximation under a variety of situations. We will refer to such a model as the “full model” as we will use it to contrast the results derived herein using the long-wave approximation. In the full model we retain the atmospheric tendencies and add back atmospheric damping in the $v$ equation. [The difference between retaining or dropping the atmospheric tendencies is negligible for the $k$ range we are interested in, but solutions are much more easily computed using Eq. (21). See Vimont (2010) or Martinez-Villalobos and Vimont (2016) for more details.] The resulting equations are projected onto the first 10 parabolic cylinder functions for consistency with the present analysis, and equatorially symmetric and antisymmetric terms are collected. The resulting equations may be written in the form of Eq. (16) with solution shown in Eq. (21), but in this case the state vector also contains the atmospheric variables.

Figure 13 shows the maximum SST growth of the first symmetric and antisymmetric optimal as a function of $\nu$ under the full model. We compare this figure with Fig. 10a that shows the same curves (symmetric and antisymmetric) but under the long-wave approximation. At first look the symmetric and antisymmetric maximum growth in both models do not look similar, but a closer look reveals more similarities than differences. The symmetric optimal does not experience growth for small $k$, and starts to grow roughly at the same value of $\nu \sim 1$. The antisymmetric optimal maximum growth is always bigger than the maximum symmetric growth (for this parameter regime), and both maximum symmetric and antisymmetric growth approach each other as $k$ increases, likely because the Kelvin wave also kills growth in the full model and its effect is reduced for shorter zonal scales.

The symmetric variability is very well approximated at small $k$ while there is an error in the antisymmetric variability even at small $k$ (the $\nu v$ term dominates the meridional momentum equation very close to the equator for antisymmetric variability at any $k$, while for the symmetric mode $\nu v$ is strictly zero at the equator and small in the vicinity of the equator). In many respects the long-wave approximated model acts like a less damped version of the full model, which explains the smaller growth shown in general in Fig. 13 compared to Fig. 10a. That parallel is not exact for the antisymmetric variability. In the full model the maximum growth occurs at $k = 0$ in agreement with Xie et al. (1999) while in the long-wave approximation the maximum growth occurs for $\nu > 0$. In spite of these differences the spatial structures of the symmetric and antisymmetric optimals (and also the time-evolved structures) for the full and long-wave approximated model look almost identical, showing that the long-wave approximation is a good representation of the WES feedback process, despite differences in the maximum growth attained (less damped than the full model).

4. Conclusions and final comments

a. Conclusions

The growth and equatorial symmetry properties of thermodynamically coupled ocean–atmosphere variability were studied using the Gill–Matsuno atmospheric model coupled to a thermodynamic slab ocean. Assuming an atmosphere completely determined by the underlying SSTs, the ocean and atmosphere variables were projected onto parabolic cylinder functions $\psi_m(y)$, effectively decomposing the variability onto different $T_m$ modes. Under the assumption of a geographically homogeneous coupling (i.e., $K_q$ and $\alpha$ constant) two independent sets of solutions emerge: equatorially symmetric ($m$ even) and equatorially antisymmetric (meridional mode–like; $m$ odd). These two sets of solution behave similarly away from the equator, for $T_m$ modes $m \geq 2$, and differ significantly in the equatorial zone, for $T_m$ modes $m = 0, 1$. The main difference in the equatorial zone is traced to the additional SST damping provided by the atmospheric Kelvin wave for equatorially symmetric variability for large zonal scales. On the other hand, antisymmetric variability close to the equator is unaffected by a damping term analogous to
the Kelvin one for symmetric variability, leading to more growth on average.

It is found that both equatorially symmetric and antisymmetric variability propagate in a similar manner toward the equator. In this framework this is realized by a decrease in amplitude of higher $T_m$ modes and an increase in amplitude of lower $T_m$ modes. When the variability reaches the equatorial zone it excites the $m = 0$ mode in the symmetric case and the $m = 1$ mode in the antisymmetric case, leading to two very different outcomes. In the symmetric case the SST signal is subjected to additional damping by the atmospheric Kelvin wave, whereas in the antisymmetric case the SST signal grows through the positive WES feedback produced by the first antisymmetric atmospheric Rossby wave. For large zonal scales the outcome of this process is growth of the antisymmetric mode, and decay of the symmetric mode. In other words, any SST distribution will grow effectively through this mechanism if its meridional structure has a large projection onto the $\psi_1(y)$ function (equatorially antisymmetric SST structure peaking around $8^\circ$ of latitude for the parameters in this study; also see Fig. A1). On the other hand an equatorially symmetric SST distribution confined to the equator [large projection onto the $\psi_0(y)$ function] will be short lived in absence of other feedbacks. For shorter zonal wavenumber, but still relatively large zonal scale, damping of symmetric structures by the Kelvin wave is reduced and the growth of both symmetric and antisymmetric variability is similar.

Under the range of parameters considered in this study the growth process of thermodynamically coupled variability is fundamentally nonnormal. This non-normality is manifest in interactions that initially grow a SST anomaly while at the same time seeding its eventual decay. Initial growth occurs as a given SST structure (without loss of generality, we consider an initial positive anomaly) produces a positive tendency for lower-order modes, and negative tendency for higher order modes. These tendencies lead to short-term growth. The positive tendency of the lower-order mode excites the next-lowest-order mode, and so forth until the lowest-order antisymmetric ($m = 1$) or symmetric ($m = 0$) structure is excited, at which that mode either grows on its own (for the $m = 1$ antisymmetric structure) or experiences enhanced decay due to the Kelvin wave (for the $m = 0$ symmetric structure, as described above). At the same time (again assuming initial positive polarity for a given mode), as the amplitude of the higher-order mode grows more large and negative, the negative amplitude of the higher-order mode in turn contributes to decay of the original SST mode. The nonnormal decay process eventually counteracts growth of the total anomaly, and is critical for maintaining stability of the system (under the parameter regime considered herein). In principle this growth process does not require equatorial antisymmetry and in fact it would prefer the symmetric structure should the Kelvin wave not exist.

Using this framework we show that both equatorially symmetric and antisymmetric modes propagate westward as a consequence of the group velocity of atmospheric Rossby waves. In the antisymmetric case the propagation is due to the zonal phasing between the antisymmetric Rossby wave wind response and SST anomalies as found by Xie et al. (1999). In the symmetric case, the interpretation is the same for $m > 2$ symmetric Rossby waves, except in the equatorial zone. In the equatorial region the propagation direction will depend on the relative phasing between the first atmospheric Rossby and Kelvin waves. It turns out that for large zonal scales, this phasing favors the growth by the first Rossby wave implying a westward propagation of the SST structure.

b. Discussion

The implications of this study, although assuming a homogeneous coupling, can be extended into a system that includes geographical variations in the mean state, as manifest by a geographic dependence of $K_q$ (Martinez-Villalobos and Vimont 2016). In that case, atmospheric heating depends on the product $K_q(y)T$, and hence the mean state acts as a “mode selector” for enhancing coupling of particular modes. As an example consider the air–sea coupling distribution provided by an equatorially symmetric but meridionally thin intertropical convergence zone (ITCZ). This coupling structure will suppress off-equatorial modes and enhance just the atmospheric Kelvin and lowest-order Rossby waves. In the limit of a vanishingly thin ITCZ [in which $K_q(y)$ approaches a delta function], coupling for antisymmetric modes are eliminated [via the product of $K_q(y)$ and antisymmetric $T_m$] and we would expect just a thermodynamically coupled symmetric mode to be able to grow, as shown in Martinez-Villalobos and Vimont (2016). Following the same idea, a broader symmetric ITCZ would produce a situation more akin to the one presented in this paper: more growth for antisymmetric modes, because the symmetric modes are additionally damped by the Kelvin wave. This interpretation could be extended to include variations in the assumed atmospheric radius of deformation as well, such as the case of a Battisti et al. (1999) or Lindzen and Nigam (1987) style model of the atmospheric boundary layer. Similar arguments could also be applied to understanding zonal evolution through a zonally and meridionally varying mean state.

This framework also helps understand the consequences of equatorial asymmetry in the coupling, for example through meridional asymmetries in the ITCZ.
position. As discussed above, a narrowly confined equatorially symmetric ITCZ (e.g., tropical Atlantic during boreal spring) suppresses coupling for off-equatorial and antisymmetric modes and enhances the atmospheric Kelvin and first symmetric Rossby responses. In contrast, for a narrow ITCZ placed off the equator (e.g., tropical Atlantic and eastern Pacific during boreal fall) the coupling is asymmetric. In that case, preference for equatorially symmetric or antisymmetric modes will depend on the relative projections of the \( \psi_1(y) \) and \( \psi_0(y) \) structures onto \( K_q(y) \). For a narrow ITCZ structure centered near a maximum in \( \psi_1(y) \), antisymmetric modes will dominate the solution [even with a nonzero projection of \( K_q(y) \) onto \( \psi_0(y) \)] due to the damping effect of the Kelvin wave for the symmetric component of the solution. In that case, however, the solution will include a mix of both symmetric and antisymmetric components. This mechanism explains the findings of Martinez-Villalobos and Vimont (2016) in that regard (cf. Figs. 4 and 5 in that paper). This interpretation may also be useful for interpreting similar variability in models with less restrictive coupling parameterizations (e.g., Lindzen and Nigam 1987; Battisti et al. 1999).

This work shows that both antisymmetric and symmetric structures emerge from the physical processes that generate tropical “meridional modes”; that is, both antisymmetric and symmetric structures grow and propagate due to the same physical processes. This is somewhat in contrast to early discussions of meridional mode behavior, in which it was thought that equatorial antisymmetry would enhance growth through providing an interhemispheric temperature gradient that would enhance the surface wind response. Given the results herein, it appears that an interhemispheric gradient is not necessary, and in fact equatorially symmetric structures would preferentially grow in the absence of the Kelvin wave. Instead, growth is more closely related to the zonal and meridional phasing between the atmospheric Rossby wave response to imposed heating and the SST that is presumably responsible for that heating in the first place (Vimont 2010). Based on these arguments, one might ask whether the term “meridional mode” is still appropriate. We show herein that both antisymmetric and symmetric structures exist as a class of structures that evolve in a similar manner toward lower meridional wavenumbers; as such, we argue that the phrase “meridional mode” is appropriate for both antisymmetric and symmetric structures.

The aim of this study is to explain the main similarities and differences between symmetric and antisymmetric thermodynamically coupled variability using a common framework. As such, besides the usual linearity assumption, there are many approximations being made and a lot of room to improve the model. For example the parameters in the model, especially the coupling, do not vary geographically. There are different regions in the tropical oceans where we would not expect the same WES feedback coupling. Another important approximation is the assumption that latent heat is unaffected by meridional wind variations. These variations will affect antisymmetric and symmetric modes differently, and we hope to consider them in a subsequent study. [The effect of mean meridional winds has been investigated by Liu and Xie (1994) and Wang (2010).] Also, the use of the long-wave approximation produces less damped modes than using a model with meridional wind damping, although that does not affect the spatial structure of the modes. Finally variations of other fluxes, especially short wave flux, should be considered in the future. Despite these caveats, this study does provide physical insight into the workings of thermodynamically coupled variability. The basic meridional mode coupling mechanism is contained within the framework here presented. We expect that the insight achieved here, by stripping down the WES interaction into its most basic building block, can be translated into explaining observed variability and more complex model output, especially slab ocean simulations.

In nature, equatorially antisymmetric modes coupled through the WES feedback are an established and relevant part of our climate system, and are the subject of a large literature collection studying meridional mode variability in both the tropical Atlantic and Pacific. On the other hand, thermodynamic equatorially symmetric variability has been less studied, likely because the Bjerknes feedback masks these kinds of modes. Nonetheless the interest in this kind of variability has increased following the findings of Clement et al. (2011). In addition the Pacific meridional mode has an important degree of equatorial symmetry (Chiang and Vimont 2004), and as such the results derived herein may be relevant in nature. We hope that this study provides a common ground to understand these modes in a unified framework.

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APPENDIX A

Parabolic Cylinder Functions

The version of the parabolic cylinder functions used in this study are given by
\[ \psi_m(y) = \frac{1}{\sqrt{2m!\pi^{1/2}}} H_m \exp(-y^2/2), \quad (A1) \]

where \( H_m \) are the Hermite polynomials. These polynomials are defined for nonnegative \( m \) integers. The first four functions are plotted in Fig. A1. The functions are orthonormal:

\[ \int_{-\infty}^{\infty} \psi_m(y) \psi_n(y) \, dy = \delta_{mn}. \quad (A2) \]

**APPENDIX B**

*Atmospheric Variables in Terms of SSTs*

We write the atmospheric variables using the parabolic cylinder functions as

\[
\begin{align*}
    u(t, x, y) &= \sum_{m=0}^{\infty} u_m(t) \psi_m(y) \exp(ikx), \\
    v(t, x, y) &= \sum_{n=0}^{\infty} v_n(t) \phi_n(y) \exp(ikx), \quad \text{and} \\
    \phi(t, x, y) &= \sum_{m=0}^{\infty} \phi_m(t) \psi_m(y) \exp(ikx). \quad (B1)
\end{align*}
\]

In solving the model equations for a particular mode \( m \) the following identity is useful:

\[
\begin{align*}
    y \psi_m &= \sqrt{\frac{m+1}{2}} \psi_{m+1} + \sqrt{\frac{m}{2}} \psi_{m-1}, \\
    \frac{d \psi_m}{dy} &= -\sqrt{\frac{m+1}{2}} \psi_{m+1} + \sqrt{\frac{m}{2}} \psi_{m-1}, \quad (B2)
\end{align*}
\]

as well as defining the auxiliary variables (Gill 1980):

\[
\begin{align*}
    u(t, y, x) &= \frac{1}{2} \left\{ q_0(t) \psi_0(y) + \sum_{m=1}^{\infty} q_{m+1}(t) \left[ \psi_{m+1}(y) - \sqrt{\frac{m+1}{m}} \psi_{m-1}(y) \right] \right\} \exp(ikx) \quad \text{and} \\
    \phi(t, y, x) &= \frac{1}{2} \left\{ q_0(t) \psi_0(y) + \sum_{m=1}^{\infty} q_{m+1}(t) \left[ \psi_{m+1}(y) + \sqrt{\frac{m+1}{m}} \psi_{m-1}(y) \right] \right\} \exp(ikx). \quad (B6)
\end{align*}
\]

The meridional wind can be found by plugging the above equations into the zonal wind and geopotential equations and solving for \( v \), yielding

\[
\begin{align*}
    v(t, y, x) &= \sum_{m=0}^{\infty} \sqrt{\frac{1}{2(m+1)}} K_2 T_{m+1}(t) \\
    &\quad + (s + ik) q_{m+1}(t) \psi_m(y) \exp(ikx). \quad (B7)
\end{align*}
\]

**APPENDIX C**

*Growth Equation*

Using Eqs. (2) and (B1) we can write the equation for the evolution of \( T_m \) and \( T^*_m \) as:

\[
q = \phi + u \quad \text{and} \quad r = \phi - u. \quad (B3)
\]

Plugging Eq. (B1) into the model Eqs. (1), and using Eqs. (2), (B2), and (B3), we find that

\[
\begin{align*}
    q_0 &= -K_2 \frac{T_0}{q} \quad \text{and} \\
    q_{m+1} &= K_2 \left[ \sqrt{m(m+1)} T_{m-1} + m T_{m+1} \right] \quad (B4)
\end{align*}
\]

In doing so we used the fact that

\[
r_{m-1} = \sqrt{\frac{m+1}{m}} q_{m+1}, \quad (B5)
\]

which is a consequence of making the long-wave approximation (Gill 1980) in the meridional wind equation. Inverting Eq. (B3) and using Eq. (B5) we get

\[
\begin{align*}
    \psi_m(y) &= \sqrt{\frac{m+1}{m}} \psi_{m+1}(y) \quad \text{and} \\
    \frac{d \psi_m}{dy} &= -\sqrt{\frac{m+1}{m}} \psi_{m+1} + \sqrt{\frac{m}{m-1}} \psi_{m-1}, \quad (B2)
\end{align*}
\]

Equations (B6) and (B7) correspond to Eq. (3) in the main text.
\[
\frac{\partial T_m}{\partial t} = \alpha u_m - \varepsilon_T T_m \quad \text{and} \quad \frac{\partial T_m^*}{\partial t} = \alpha (u_m T_m^* + u^*_m T_m) - 2\varepsilon |T_m|^2. \tag{C2}
\]

Multiplying the first equation by \(T_m^*\) and the second one by \(T_m\) and adding them we get

\[
\frac{\partial |T_m|^2}{\partial t} = K_q \alpha \left\{ \frac{1}{(2m-1)^2\varepsilon^2 + k^2} [(m-1)(2m-1)\varepsilon]|T_m|^2 \\
+ \frac{1}{(2m+3)^2\varepsilon^2 + k^2} [(m+2)(2m+3)\varepsilon]|T_m|^2 \\
- \frac{1}{(2m-1)^2\varepsilon^2 + k^2} [\sqrt{m(m-1)}(2m-1)\varepsilon]|T_m||T_{m-2}| \cos(\delta_m - \delta_{m-2}) \\
+ \frac{1}{(2m+3)^2\varepsilon^2 + k^2} [\sqrt{(m+2)(m+1)}(2m+3)\varepsilon]|T_m||T_{m+2}| \cos(\delta_m - \delta_{m+2}) \\
- \frac{1}{(2m-1)^2\varepsilon^2 + k^2} [\sqrt{m(m-1)}k]|T_m||T_{m-2}| \sin(\delta_m - \delta_{m-2}) \\
+ \frac{1}{(2m+3)^2\varepsilon^2 + k^2} [\sqrt{(m+2)(m+1)}k]|T_m||T_{m+2}| \sin(\delta_m - \delta_{m+2}) \right\} - 2\varepsilon_T |T_m|^2. \tag{C3}
\]

Here \(\delta_m - \delta_{m-2}\) is the phasing in the complex plane between modes \(m\) and \(m-2\), and comes from the decomposition \(T_m = |T_m| \exp(i\delta_m)\).

To find the equation for total SST growth we use Eq. (2) and write the total SST amplitude squared \(|T|^2\) in terms of the individual modes:

\[
|T|^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} T_m^* T_n \psi_m \psi_n. \tag{C4}
\]

Integrating \(|T|^2\) over the whole domain (the angle brackets denote that integration) we get the total growth that can be decomposed in terms of the growth of individual terms as follows:

\[
\langle |T|^2 \rangle = \int_{-L_x}^{L_x} \int_0^\infty |T|^2 \, dx \, dy = L_x \sum_{m=0}^{\infty} |T_m|^2, \tag{C5}
\]

where \(L_x\) is the zonal extension of the basin considered and we have used Eq. (A2). Using Eq. (C3) the equation for the total SST growth is given by

\[
\frac{\partial}{\partial t} \langle |T|^2 \rangle = L_x K_q \alpha \sum_{m=0}^{\infty} \left\{ \frac{1}{(2m-1)^2\varepsilon^2 + k^2} [(m-1)(2m-1)\varepsilon]|T_m|^2 \\
+ \frac{1}{(2m+3)^2\varepsilon^2 + k^2} [(m+2)(2m+3)\varepsilon]|T_m|^2 \\
+ \frac{2}{(2m+3)^2\varepsilon^2 + k^2} [\sqrt{(m+2)(m+1)}k]|T_m||T_{m+2}| \sin(\delta_m - \delta_{m+2}) \right\} \\
- 2L_x \varepsilon_T \sum_{m=0}^{\infty} |T_m|^2. \tag{C6}
\]
Notice there is a partial cancellation between the exchange terms.

**APPENDIX D**

**Growth Optimization**

To find the initial conditions that maximize SST growth by the WES feedback at some time later \( \tau \) we use a standard Lagrangian multiplier approach. The function \( \mathcal{L} \) contains the information to optimize:

\[
\mathcal{L} = |T(\tau)|^2 - \mu |T(0)|^2 - c. \tag{D1}
\]

Here \( T(\tau) \) corresponds to the vector \( [T_0(\tau), T_1(\tau), T_2(\tau), \ldots] \). The first term on the right-hand side \( |T(\tau)|^2 = \sum_{m=0}^{\infty} |T_m(\tau)|^2 \) (cf. Eq. (C5)); we are optimizing \((1/L_s)\lambda(|T(\tau)|^2)\) is the quantity to be optimized, while the second term is the constraint (i.e., initial condition amplitude equal to some arbitrary constant \( c \)). Using Eq. (21) and taking the derivative of the previous equation respect to \( T(0) \), we find that the initial condition \( T(0) \) that maximizes growth at time \( \tau \) satisfies the following eigenvalue problem:

\[
G(\tau)G(\tau)T(0) = \mu T(0), \tag{D2}
\]

where \( \mu \) corresponds to the growth at time \( \tau \).

**REFERENCES**


