Structural Bifurcation of 2-D Incompressible Flows

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This paper is the authors' tribute to the mathematical and human contributions of Ciprian Foias and Roger Temam

ABSTRACT. We study in this article the structural bifurcation of divergence-free vector fields on a two-dimensional (2-D) compact manifold M. We prove that, for a one-parameter family of divergence-free vector fields $u(\cdot, t)$ structural bifurcation—i.e., change in their topological-equivalence class—occurs at t_0 if $u(\cdot, t_0)$ has a degenerate singular point $x_0 \in \partial M$ such that $\partial u(x_0, t_0)/\partial t \neq 0$. Careful analysis of the trajectories allows us to give a complete classification of the orbit structure of $u(x, t)$ near (x_0, t_0). This article is part of a program to develop a geometric theory for the Lagrangian dynamics of 2-D incompressible fluid flows.

1. INTRODUCTION

M. G. first met Ciprian while he was still a graduate student at the Courant Institute in New York and Roger during a sabbatical at the Ecole Normale Supérieure in Paris. He has learned a lot from their publications and even more from conversations with them. S. W. has been very fortunate to be part of the Foias-Temam school of fluid dynamics since the early 1990s. Both Ciprian and Roger's scientific work has been a constant source of inspiration for him. The authors hope that this paper is in Ciprian's and Roger's spirit of applying advanced mathematical concepts and methods to problems in fluid mechanics and other areas of the physical sciences.

Fluid flows can be described by the Eulerian approach or the Lagrangian approach (cf. Batchelor [3]). In the Eulerian approach, the field of motion is described as a function of spatial coordinates and time, and is governed by a set of partial differential equations (PDEs) such as the Euler or the Navier-Stokes equations (see Caffarelli, Kohn and Nirenberg [6], Constantin and Foias [7], Lions [11], Majda [17], Temam [26]). In the Lagrangian approach, the structure of the flow in the physical space that the fluid occupies is described by the particle's trajectories, particles being tagged by their original positions; this leads to an infinite set of ordinary differential equations. A fundamental mathematical and physical question is to study the structure of the solutions of the partial differential equations for incompressible fluid flows in the physical space. In this case, the Lagrangian flow description is volume preserving in the physical space.

The most fundamental questions about such incompressible, or divergence-free, flows are topological. The use of topological ideas in physics and fluid mechanics goes back to the very origin of topology as an independent science. A new impetus was given to the application of topological methods in fluid mechanics by the pioneering work of V. I. Arnold [1]; see also the recent book by Arnold and Khesin [2] for various topological aspects of hydrodynamics, and Brenier [4, 5] for volume-preserving maps and generalized solutions of the Euler equations.

We study in this article the structural bifurcation, or change of topological-equivalence class, of one-parameter families of divergence-free vector fields on a two-dimensional (2-D) compact manifold with or without boundary. The main motivation of the study comes from the fact that the solutions of the PDEs governing incompressible fluid flows—such as the Euler equations and the Navier-Stokes equations—can be viewed as one-parameter families of divergence-free vector fields with the time t as the parameter. Similar remarks apply to the quasi-geostrophic equations that govern rotating flows, such as large-scale atmospheric circulations on the sphere or oceanic circulations in a bounded domain (Ghil and Childress [8], Lions, Temam and Wang [12, 13], Pedlosky [19]).

The results reported here use only the smoothness and the divergence-free character of the vector field families under study; consequently, they are potentially applicable to many physical problems of this nature. The results and methods given in the present article show precisely *how* a structural transition occurs. An important open question, however, is to predict *which* structural bifurcation would occur for a given time-dependent solution of the Euler, Navier-Stokes, or quasi-geostrophic equations of fluid flow. For instance, one needs to locate the degenerate singular point x_0 that gives rise to the bifurcation, the index of x_0, and the time t_0 at which the bifurcation will arise. We plan to address such questions in future work, as part of a program to develop systematically a geometric theory for the Lagrangian dynamics of 2-D incompressible flows, whether rotating or not. Observational and numerical evidence for the relevance of our topological results to various 2-D incompressible fluid flows encourages us to pursue such a program.

This being said, consider a one-parameter family of divergence-free vector fields $u(x,t)$, where $x \in M$ stands for a point on the 2-D manifold M with boundary ∂M, and $t \in [0,T]$ is the parameter. A natural method for studying structural bifurcations of such a family $u(x,t)$ is to Taylor expand it near t_0, and then to analyze its structure in a neighborhood of t_0 by taking the first-order approximation,

$$
\begin{cases}
u(x,t) = u^0(x) + (t - t_0)u^1(x) + v(x, t - t_0), \\
v(x, t - t_0) = o(|t - t_0|), \\
u^0(x) = u(x, t_0), \\
u^1(x) = \dfrac{\partial u(x,t)}{\partial t}\Big|_{t=t_0}.
\end{cases}
\tag{1.1}
$$

The main results of this article are Theorems 4.1–4.3. First the orbit structure of $u(x,t)$ near (x_0, t_0) is fully classified in Theorems 4.1 and 4.2. Then we prove in Theorem 4.3 that, for a one-parameter family of C^1 divergence-free vector fields, structural bifurcation occurs at t_0 if the two conditions (4.1) and (4.2) in Section 4 are satisfied. Roughly speaking, the first condition (4.1) requires $x_0 \in \partial M$ to be an isolated and degenerate singular point of u^0. This condition is necessary for structural bifurcation, according to the structural stability theorem of Ma and Wang [14, 15, 16]. The second condition (4.2) requires that the first-order term u^1 not be zero at x_0, i.e., $u^1(x_0) \neq 0$. This condition suffices to drive the flow change structure when t crosses the singular point t_0. Otherwise, one needs to study higher-order terms in the Taylor expansion (1.1).

To classify the orbit structure of $u(x,t)$ near the singular point x_0 for t close to t_0, we establish a full classification of isolated and degenerate singular points of a divergence-free vector field in Theorem 3.2. The use of this classification theorem extends to other situations as well. Thanks to this theorem, it is easy to see that condition (4.1) is equivalent to the index of u^0 at x_0 being different from $-\frac{1}{2}$. Based on the information about ind (u^0, x_0), Theorems 4.1 and 4.2 classify the flow structure of $u(x,t)$ near (x_0, t_0).

We believe that this classification, as well as the technical methods developed in the present article to prove the theorems, shall be useful in analyzing boundary separations of actual fluid flows. The nature of a flow's separation from the boundary plays a fundamental role in many physical problems, and often determines the nature of the flow in the interior as well (Haidvogel et al. [9] and Jiang et al. [10]).

This article is organized as follows. Notation and preliminaries are described in Section 2. Necessary conditions for structural bifurcation and the Singularity Classification Theorem are obtained in Section 3. Section 4 states the main theorems and gives some illustrative examples. The theorems are proved in Section 5 via a few technical lemmas; the key technical methods involved, which could be useful in other problems, appear in these lemmas.

2. Preliminaries

Let $M \subset \mathbb{R}^2$ be a closed and bounded domain with C^{r+1} $(r \geq 2)$ boundary ∂M. All results in this article hold true when M is a 2-D compact manifold with boundary, which is diffeomorphic to a submanifold of the unit sphere S^2, or to S^2 itself.

Let TM be the tangent bundle of M, and $C^r(TM)$ be the space of all C^r vector fields on M. We consider two subsets of $C^r(TM)$:

$$D^r(TM) = \{v \in C^r(TM) : v_n|_{\partial M} = 0,\ \operatorname{div} v = 0\},$$
$$B^r(TM) = \left\{v \in D^r(TM) : \frac{\partial v_\tau}{\partial n}\Big|_{\partial M} = 0\right\};$$

here $v_n = v \cdot n$ and $v_\tau = v \cdot \tau$, while n and τ are the unit normal and tangent vectors on ∂M respectively. It is easy to see that $B^r(TM) \subset D^r(TM)$. The boundary condition $v_n = 0$ on ∂M is the so-called *free-slip* boundary condition used for the Euler equations (e.g., Temam [26] or Truesdell [27]); this boundary condition is also used for certain models of the large-scale ocean circulation (see Jiang et al., [10]). For the Navier-Stokes equations and other models of the ocean circulation (Haidvogel et al. [9]), one uses the so-called *no-slip* boundary conditions, where $v = 0$ on ∂M is required. The vector fields in $B^r(TM)$ satisfy both $v \cdot n = 0$ and $\partial_n v_\tau = 0$ on ∂M, which is also commonly used in ocean circulation models. The results obtained in this article can be generalized to no-slip boundary conditions as well, by using different techniques, and will be reported elsewhere.

Definition 2.1. Two vector fields $u,\ v \in D^r(TM)$ are called topologically equivalent if there exists a homeomorphism of $\varphi : M \to M$, which takes the orbits of u to orbits of v and preserves their orientation.

We know that the solutions of the Euler or Navier–Stokes equations of 2-D incompressible fluid flow are one-parameter families of divergence-free vector fields [3, 7, 26]. Similar results hold for the quasi-geostrophic equations that govern planetary-scale rotating flows [8, 12, 13, 19]. This motivates the present study of the structural bifurcation of such families on a 2-D compact manifold with boundary, in view of applying the results to the 2-D Euler, Navier-Stokes or quasi-geostrophic equations of incompressible flow.

We start with some basic concepts. Let $X = D^r(TM)$ or $B^r(TM)$ in the following three definitions.

Definition 2.2. Let $u \in C^1([0,T], X)$. The vector field $u_0 = u(\cdot, t_0)$ $(0 < t_0 < T)$ is called a bifurcation point of u at time t_0 if, for any $t^- < t_0$ and $t_0 < t^+$ with t^- and t^+ sufficiently close to t_0, the vector field $u(\cdot; t^-)$ is not topologically equivalent to $u(\cdot; t^+)$. In this case, we say that $u(x,t)$ has a bifurcation at t_0 in its global structure.

Definition 2.3. Let $u \in C^1([0,T], X)$. We say that $u(x,t)$ has a bifurcation in its local structure in a neighborhood $U \subset M$ of x_0 at t_0 $(0 < t_0 < T)$ if, for any $t^- < t_0$ and $t_0 < t^+$ with t^- and t^+ sufficiently close to t_0, the vector fields $u(\cdot; t^-)$ and $u(\cdot; t^+)$ are not topologically equivalent locally in $U \subset M$.

Definition 2.4. A vector field $v \in X$ is called structurally stable in X if there exists a neighborhood $\mathcal{O} \subset X$ of v such that for any $u \in \mathcal{O}$, u and v are topologically equivalent.

We recall next some basic facts and definitions on divergence-free vector fields. Let $v \in D^r(TM)$.

1. A point $p \in M$ is called a singular point of v if $v(p) = 0$; a singular point p of v is called non-degenerate if the Jacobian matrix $Dv(p)$ is invertible; v is called regular if all singular points of v are non-degenerate.
2. An interior non-degenerate singular point of v can be either a center or a saddle, and a non-degenerate boundary singularity must be a saddle.
3. Saddles of v must be connected to saddles. An interior saddle $p \in \overset{\circ}{M}$ is called *self-connected* if p is connected only to itself, i.e., p occurs in a graph whose topological form is that of the number 8.
4. v is structurally stable near each non-degenerate singular point of v.

The following theorem was proved in [14, 15, 16], providing necessary and sufficient conditions for structural stability of a divergence-free vector field.

Theorem 2.1. *Let $X = D^r(TM)$ or $X = B^r(TM)$, $r \geq 1$, and $v \in X$. Then v is structurally stable in X if and only if*

(1) *v is regular;*
(2) *all interior saddles of v are self–connected; and*
(3) *each boundary saddle point is connected to boundary saddle points on the same connected component of the boundary.*

Moreover, the set of all structurally stable vector fields is open and dense in X.

This theorem provides necessary and sufficient conditions for structural stability of a divergence-free vector field. The study of structural stability has been the main driving force behind much of the development of dynamical systems theory (see among others [18, 20, 21, 22, 23, 24, 25]). We are interested here in the structural stability of a 2-D divergence-free vector field subject to perturbations by divergence-free vector fields. Notice that the divergence-free condition changes completely the general features of structurally stable fields as compared to the situation when this condition is not present. The latter case was studied in 2-D by Peixoto [20]. The conditions for structural stability and genericity in Peixoto's theorem are: (i) the field can have only a finite number of singularities and closed orbits (critical elements) which must be hyperbolic; (ii) there are no saddle connections; (iii) the non-wandering set consists of singular points and closed orbits.

The first condition in Theorem 2.1 above requires only regularity of the field and does not exclude centers; the latter are not hyperbolic and thus are excluded by condition (i) in Peixoto's result. Our theorem's second condition is also of a completely different nature than the corresponding one in the Peixoto theorem. Namely, Peixoto's condition (ii) excludes the possibility of saddle connections altogether, while our condition (2) requires all interior saddles be self-connected!

3. NECESSARY CONDITIONS FOR STRUCTURAL BIFURCATION

By Theorem 2.1 and item (4) before Theorem 2.1, we obtain easily the following necessary conditions for the appearance of structural bifurcation for one-parameter divergence-free vector fields.

Theorem 3.1. *Let $u \in C^1([0,T], D^r(TM))$, $r \geq 1$.*

(1) *If $u(x,t)$ has a bifurcation in its local structure in a neighborhood $U \subset M$ of x_0 at t_0 ($0 < t_0 < T$), then x_0 must be a degenerate singular point of $u(x,t)$ at t_0.*
(2) *If $u(x,t)$ has a bifurcation in its global structure at t_0 ($0 < t_0 < T$), then $u(x,t_0)$ does not satisfy one or more of the conditions 1–3 for structural stability of Theorem 2.1.*

This theorem suggests studying the structure of divergence-free vector fields near degenerate singular points. For this purpose, we recall the definition of indices of singular points of a vector field. Let $p \in M$ be an isolated singular point of $v \in C^r(TM)$; then

$$\mathrm{ind}(v,p) = \deg(v,p),$$

where $\deg(v,p)$ is the Brouwer degree of v at p.

Let $p \in \partial M$ be an isolated singular point of v, and $\widetilde{M} \subset \mathbb{R}^2$ be an extension of M, i.e., $M \subset \widetilde{M}$ such that $p \in \widetilde{M}$ is an interior point of $\widetilde{M}$. In a neighborhood of p in $\widetilde{M}$, v can be extended by reflection to $\tilde{v}$ such that p is an interior singular point of $\tilde{v}$, thanks to $v \cdot n|_{\partial M} = 0$, the condition of no normal flow. Then we define the index of v at $p \in \partial M$ by

$$\mathrm{ind}(v,p) = \frac{1}{2}\,\mathrm{ind}(\tilde{v},p). \tag{3.1}$$

Let $p \in M$ be an isolated singular point of $v \in C^r(TM)$. An orbit γ of v is said to be a stable orbit (resp. an unstable orbit) connected to p, if the limit set $\omega(x) = p$ (resp. $\alpha(x) = p$) for any $x \in \gamma$.

We now introduce a singularity classification theorem for incompressible vector fields, which will be useful in our discussion of structural bifurcation.

Theorem 3.2 (Singularity Classification Theorem)**.** *Let M be a 2-D compact manifold with or without boundary, and $p \in M$ be an isolated singular point of $v \in D^r(TM)$, $r \geq 1$. Then p is connected only to a finite number of orbits and the stable and unstable orbits connected to p alternate when tracing a closed curve around p. Furthermore*

(1) *when $p \in \overset{\circ}{M}$, p has $2n$ ($n \geq 0$) orbits, n of which are stable, and the other n unstable, while the index of v at p is:*

$$\mathrm{ind}(v,p) = 1 - n; \tag{3.2}$$

(2) *when $p \in \partial M$, p has $n+2$ ($n \geq 0$) orbits, two of which are on the boundary ∂M, and the index of p is:*

$$\text{ind}(v, p) = -\frac{n}{2}. \tag{3.3}$$

Proof. We proceed in a few steps.

STEP 1 Let Γ be the set of all orbits connected to p. Since the flow Φ of v is area-preserving, it is easy to see that Γ has no interior point, that is, $\mathring{\Gamma} = \emptyset$. Assuming otherwise, we shall derive a contradiction.

Let $x \in \mathring{\Gamma}$; then there exists a neighborhood $\mathcal{O} \subset M$ of x_0 such that $\mathcal{O} \subset \mathring{\Gamma}$. Without loss of generality, we assume that

$$\tilde{\mathcal{O}} = \{x \in \mathcal{O} \mid \omega(x) = p\}$$

has positive measure. Then

$$\lim_{t \to \infty} |\Phi(\tilde{\mathcal{O}}, t)| = 0,$$

a contradiction to v being incompressible.

STEP 2 We shall show that Γ has only a finite number of orbits. Assuming otherwise, there is a sequence of orbits $\gamma_j \subset \Gamma$ ($j = 1, 2, \ldots$) and an orbit orbit $\gamma_0 \subset \Gamma$ such that $\gamma_j \to \gamma_0$ in a neighborhood U of p. Without loss of generality, we assume that γ_0 is a stable orbit of p. Take $x \in \gamma_0 \cap U$, an arc Σ transversal to γ_0 at x, and N_0 sufficiently large such that Σ is transversal to γ_j for all $j \geq N_0$. If all γ_j ($j \geq N_0$) are stable orbits of p, i.e., there is no unstable orbit of p in a neighborhood of γ_0, then there must exist a neighborhood Σ_0 of x in Σ such that $\omega(z) = p$ for any $z \in \Sigma_0$, which implies that $\mathring{\Gamma} \neq \emptyset$, a contradiction.

Assume that there is a subsequence γ_{j_k} of γ_j ($j_k \geq N_0$) such that each γ_{j_k} ($1 \leq k$) is an unstable orbit of p. Let $x_{j_k} = \gamma_{j_k} \cap \Sigma$; then by $\gamma_{j_k} \to \gamma_0$ ($k \to \infty$), $x_k \to x_0$. Since the flow leaves p on the orbits γ_{jk}, and approaches p on γ_0, x_0 is a singular point. Since $x_0 \in \gamma_0 \cap U$ is taken arbitrarily, any point in $\gamma_0 \cap U$ is a singular point, a contradiction to p being an isolated singular point. Therefore p is connected to only a finite number of orbits.

STEP 3 Let γ_1, γ_2 be two adjacent orbits of v connected to p. We take an arc Σ transversal to v and intersecting with γ_1 and γ_2 at x_1 and x_2, respectively, such that there are no singular points of v and orbits of v connected to p in the domain D enclosed by γ_1, γ_2, and the arc Σ; see Figure 3.1. Let Q_1, Q_2 be sufficiently small neighborhoods of x_1 and x_2 in D. If both γ_1 and γ_2 are stable orbits of p, then for any $z \in Q_1$ ($z \neq x_1$), the orbit $\Phi(z, t) \subset D$ for $0 \leq t < t_0$, and $z_0 = \Phi(z, t_0) \in \Sigma$, but $z_0 \notin \bar{Q}_2$, which means that there is an unstable orbit of p in D, a contradiction. Therefore if γ_1 and γ_2 are adjacent and γ_1 is stable (or unstable), then γ_2 must be unstable (or stable).

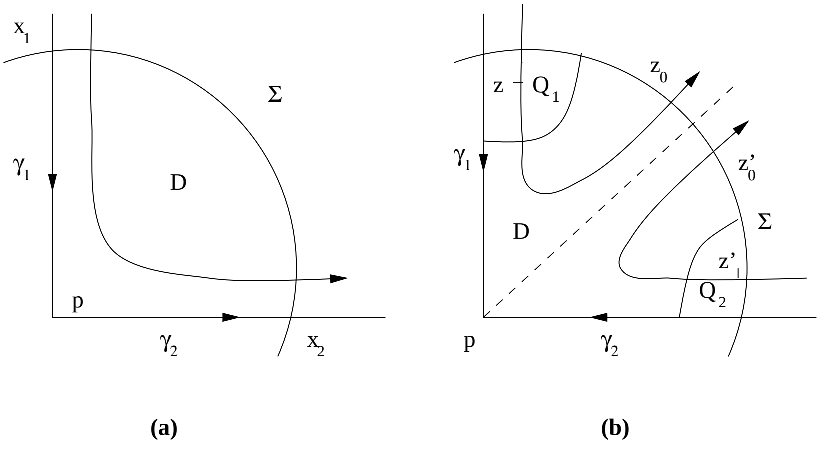

FIGURE 3.1

Since the stable and unstable orbits of a singular point p are placed adjacent to each other, it is easy to see that when $p \in \overset{\circ}{M}$, p must be connected to $2n$ $(n \geq 0)$ orbits, n of which are stable and the rest n of which are unstable. Moreover, when $p \in \partial M$ is a singular point, p must be connected to two orbits on ∂M and p must be connected to n $(n \geq 0)$ orbits in the interior of M.

STEP 4 Finally, by the Brouwer degree theory, it is well known that if $p \in \overset{\circ}{M}$ is an isolated singular point of a vector field $v \in C^r(TM)$ $(r \geq 0)$ connected to $2n$ $(n \geq 0)$ orbits, then the index of p is $1-n$, i.e., (3.2) holds true.

Let $p \in \partial M$ have $n+2$ $(n \geq 0)$ orbits. Then we can see that for the vector field $\tilde{v}$ obtained by reflective extension vector field of v on the extension manifold $\widetilde{M}$ of M, $p \in \widetilde{M}$ is an isolated interior singular point of $\tilde{v}$, which has $2n+2$ orbits. Hence (3.3) follows from (3.1) and (3.2), and the proof is complete. ◻

4. STRUCTURAL BIFURCATION ON SINGULAR BOUNDARY POINTS

Let $M \subset \mathbb{R}^2$ be a C^r $(r \geq 1)$ 2-D compact manifold with boundary, and X be either $D^r(TM)$ or $B^r(TM)$. Let $u \in C^1([0,T],X)$ be a one-parameter family of divergence-free vector fields, and be Taylor expanded by (1.1), with leading terms $u^0(x)$ and $(t-t_0)u^1(x)$ as defined there. To investigate the structural bifurcation of $u(x,t)$ at t_0, under proper conditions on these leading terms of the Taylor expansion, it suffices to consider the topological structure of the vector fields $u^0 \pm \varepsilon u^1$ for $\varepsilon > 0$ sufficiently small.

We start with some natural conditions. Assume that $u(x,t)$ satisfies the following two conditions:

$$\begin{cases} \text{there exists an isolated degenerate singular point } x_0 \in \partial M \\ \text{of } u^0\text{, and there exists some } k \geq 2 \text{ such that } u^0,\ u^1 \in C^{k+1} \\ \text{near } x_0\text{, and } \partial^k(u^0_\tau(x_0))/\partial\tau^k \neq 0; \end{cases} \tag{4.1}$$

$$u^1(x_0) \neq 0. \tag{4.2}$$

Here $v_\tau = v \cdot \tau$, and τ is the unit tangent vector on ∂M. A few remarks are now in order.

Remark 4.1. As we mentioned in the Introduction, condition (4.2) is a natural condition. If $u^1(x_0) = 0$, then one needs to consider higher-order terms in the Taylor expansion (1.1), and similar results will still be true.

Remark 4.2. Condition (4.1) is equivalent to the existence of an isolated singular point $x_0 \in \partial M$ of u_0 such that

(a) x_0 is a degenerate singular point, and

(b) $\partial^k(u^0_\tau(x_0))/\partial\tau^k \neq 0$.

By Theorem 3.1, condition (a) is a necessary condition for the structural bifurcation of $u(x,t)$ at t_0; it amounts to saying that

$$\text{ind}(u^0, x_0) \neq -\frac{1}{2},$$

thanks to our Singularity Classification Theorem in the previous section.

Remark 4.3. Let k be the smallest integer satisfying condition (b) in the previous remark. It is obvious that $k \geq 2$ since x_0 is a degenerate singular point, and hence $\partial(u^0_\tau(x_0))/\partial\tau = 0$. Therefore conditions (b) and (4.2) imply that there exists a neighborhood $\Gamma \subset \partial M$ of x_0 such that for any $z \in \Gamma$, $z \neq x_0$,

$$\frac{d}{d\tau}\left(\frac{|u^0(z)|}{|u^1(z)|}\right) \neq 0. \tag{4.3}$$

Condition (4.3) avoids rapid oscillation of the vector field $u(x,t)$ near x_0. It will be used in the proof of Lemma 5.1; see (5.6).

The main results of this article are the following theorems.

Theorem 4.1. *Let $u \in C^1([0,T], X)$ be a one-parameter family of divergence-free vector fields satisfying* (4.1) *and* (4.2). *Then in a neighborhood $\Gamma \subset \partial M$ of x_0, the singular points of $u(x, t_0 \pm \varepsilon)$ are non-degenerate for any $\varepsilon > 0$ sufficiently small. Moreover, for any $\varepsilon > 0$ sufficiently small,*

(1) *if the index* $\text{ind}(u^0, x_0)$ *is not an integer, then each of $u(x, t_0 \pm \varepsilon)$ has only one singular point on $\Gamma \subset \partial M$; and*

(2) *if* $\operatorname{ind}(u^0, x_0)$ *is an integer, then one of* $u(x, t_0 \pm \varepsilon)$ *has two singular points on* Γ*, and the other one has no singular points on* Γ.

Theorem 4.2. *Let* $u \in C^1([0,T], X)$ *satisfy* (4.1) *and* (4.2). *Then in a neighborhood* $V \subset M$ *of* x_0*, for any* $\varepsilon > 0$ *sufficiently small, each isolated interior singular point of* $u(x, t_0 \pm \varepsilon)$ *is either a center* (*with index* 1) *or a saddle* (*with index* -1).

Theorem 4.3 (Structural Bifurcation Theorem). *Let* $u \in C^1([0,T], X)$ *satisfy* (4.1) *and* (4.2). *The following assertions hold true:*

(1) $u(x,t)$ *has a bifurcation in its local structure at* (x_0, t_0);
(2) *If* $x_0 \in \partial M$ *is a unique degenerate singular point of* u^0 *on* ∂M*, then* $u(x,t)$ *has a bifurcation in its global structure at* $t = t_0$.

Before we give the proofs of these theorems, we proceed first with some examples that illustrate how structural bifurcation occurs in some prototype situations.

Example 4.4. When $x_0 \in \partial M$ is a singular point of $u^0(x) = u(x, t_0)$ with $\operatorname{ind}(u^0, x_0) = 0$, the bifurcation occurs as shown in Figures 4.1(a)–(c). Here u^0 is given by Figure 4.1(b). For any small $\varepsilon > 0$, $u(x, t_0 - \varepsilon)$ given by Figure 4.1(a) has no singular points near x_0. On the other hand, $u(x, t_0 + \varepsilon)$ given by Figure 4.1(c) has two singular points near x_0 on the boundary and one center near x_0 in the interior, all of which are non-degenerate. Hence, structural bifurcation does occur locally. As indicated by Theorem 4.3, if $x_0 \in \partial M$ is a unique degenerate singular point of u^0 on ∂M, then the structural bifurcation of $u(x,t)$ at $t = t_0$ is global.

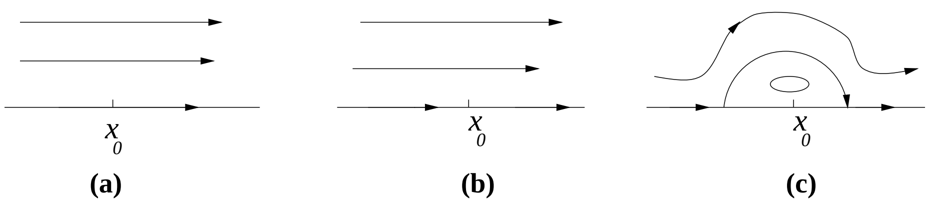

FIGURE 4.1

Example 4.5. For the case where $\operatorname{ind}(u^0, x_0) = -1$, by Theorem 3.2 on page 164, there are two orbits of u^0 connected to $x_0 \in \partial M$; such a vector field u^0 is given schematically by Figure 4.2(b). The structural evolution of $u(x,t)$ in a neighborhood of x_0 is as shown in Figures 4.2(a)–(c). Due to condition (4.2), for any small $\varepsilon > 0$, $u(x, t_0 - \varepsilon)$ given by Figure 4.2(a) has no singular points near x_0 again, while $u(x, t_0 + \varepsilon)$ given by Figure 4.2(c) has two singular points near x_0 on the boundary, both of which are non-degenerate.

Example 4.6. When $\operatorname{ind}(u^0, x_0) = -\frac{3}{2}$, there are five orbits of u^0 connected to $x_0 \in \partial M$, two of which lie on ∂M, and the other three in $\mathring{M}$. The structural

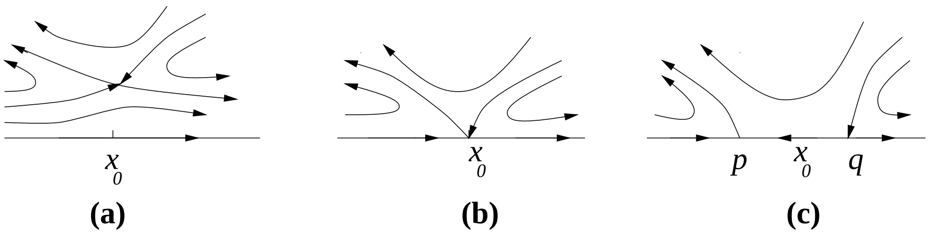

FIGURE 4.2

evolution of $u(x,t)$ in a neighborhood of x_0 is described by Figures 4.3(a)–(c). The two examples 4.5 and 4.5 correspond to the case (2) of Theorem 4.1, while the present example corresponds to its case (1).

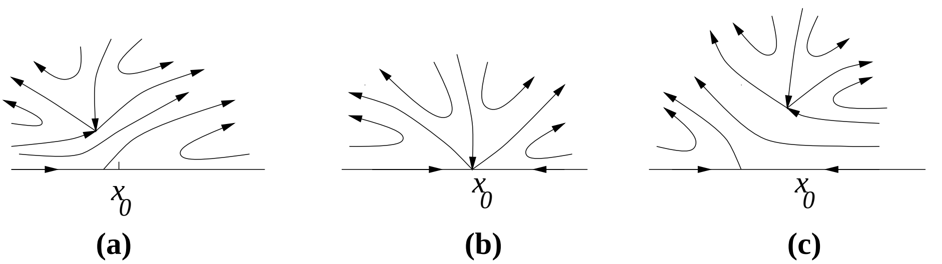

FIGURE 4.3

5. Proof of the Main Theorems

5.1. Proof of Theorem 4.1. The proof of the theorem is achieved in a few lemmas as follows.

Lemma 5.1. *Assume* (4.1) *and* (4.2). *Then there exists a small neighborhood* $\Gamma \subset \partial M$ *of* x_0 *such that all singular points of* $u^0 \pm \varepsilon u^1$ *on* Γ *are non-degenerate for any* $\varepsilon > 0$ *sufficiently small.*

Proof. Take a small neighborhood $\Gamma \subset \partial M$ of x_0 such that (4.3) holds true for any $z \in \Gamma$, $z \neq x_0$. Let $z_0 \in \Gamma \subset \partial M$ such that $z_0 \neq x_0$ and

$$u^0(z_0) \pm \varepsilon u^1(z_0) = 0, \quad \varepsilon > 0. \tag{5.1}$$

Let (x_1, x_2) be an orthogonal coordinate system with the origin at z_0, x_1–axis tangent to ∂M at z_0, and with the x_2–axis pointing inward in the normal direction. Near x_0, the vector fields u^i $(i = 0, 1)$ are expressed as $u^i = (u^i_1(x_1, x_2), u^i_2(x_1, x_2))$.

Let $v(x) = (v_1(x), v_2(x))$ be any vector field defined near x_0 such that $v \cdot n|_{\partial M} = 0$; consequently $v_2(0) = 0$ in these local coordinates. We have

$$\begin{aligned}\frac{\partial v_2(0)}{\partial x_1} &= \lim_{\substack{\Delta x=(\Delta x_1,\Delta x_2)\to 0\\ \Delta x\in\partial M}} \frac{v_2(\Delta x_1,\Delta x_2) - v_2(0)}{\Delta x_1}\\ &= \lim_{\substack{\Delta x=(\Delta x_1,\Delta x_2)\to 0\\ \Delta x\in\partial M}} \frac{v_\tau(\Delta x_1,\Delta x_2)\sin(\tau, x_1)}{\Delta x_1}.\end{aligned}$$

Here τ is a unit tangent vector at $\Delta x \in \partial M$, $v_\tau = v \cdot \tau$, and

$$\lim_{\substack{\Delta x=(\Delta x_1,\Delta x_2)\to 0\\ \Delta x\in\partial M}} \frac{\sin(\tau, x_1)}{\Delta x_1} = k(0),$$

$$\lim_{\substack{\Delta x=(\Delta x_1,\Delta x_2)\to 0\\ \Delta x\in\partial M}} v_\tau(\Delta x) = v_1(0),$$

where $k(0)$ is the curvature of ∂M at $x = 0$. Hence

$$\frac{\partial v_2(0)}{\partial x_1} = v_1(0)k(0). \tag{5.2}$$

We infer then from (5.1) and (5.2) that

$$\frac{\partial(u_2^0(0) \pm \varepsilon u_2^1(0))}{\partial x_1} = k(0)(u_1^0(0) \pm \varepsilon u_1^1(0)) = 0.$$

Consequently, the Jacobian matrix of $u^0 \pm \varepsilon u^1$ at $x = 0$ (i.e., z_0) is

$$D(u^0 \pm \varepsilon u^1)(0) = \begin{pmatrix} \dfrac{\partial u_1^0}{\partial x_1} \pm \varepsilon \dfrac{\partial u_1^1}{\partial x_1} & * \\ 0 & \dfrac{\partial u_2^0}{\partial x_2} \pm \varepsilon \dfrac{\partial u_2^1}{\partial x_2} \end{pmatrix}_{x=0}.$$

Thanks to the non-divergence of $u(x,t)$, it is easy to see that

$$\frac{\partial u_1^0(0)}{\partial x_1} \pm \varepsilon \frac{\partial u_1^1(0)}{\partial x_1} = -\left[\frac{\partial u_2^0(0)}{\partial x_2} \pm \varepsilon \frac{\partial u_2^1(0)}{\partial x_2}\right].$$

Therefore it suffices to prove that

$$\frac{\partial u_1^0(0)}{\partial x_1} \pm \varepsilon \frac{\partial u_1^1(0)}{\partial x_1} \neq 0. \tag{5.3}$$

Again, since the x_1–axis is tangent to ∂M at $x = 0$,

$$\frac{\partial u_1^0(0)}{\partial x_1} = \frac{d}{d\tau}|u^0(z_0)|, \quad \frac{\partial u_1^1(0)}{\partial x_1} = \frac{d}{d\tau}|u^1(z_0)|, \tag{5.4}$$

where $|u^i| = \sqrt{|u_1^i|^2 + |u_2^i|^2}$. By (5.1),

$$\varepsilon = \mp\frac{|u^0(z_0)|}{|u^1(z_0)|}. \tag{5.5}$$

From (5.4) and (5.5), we obtain:

$$\frac{\partial u_1^0(0)}{\partial x_1} \pm \varepsilon\frac{\partial u_1^1(0)}{\partial x_1} = \frac{d}{d\tau}|u^0(z_0)| - \frac{|u^0(z_0)|}{|u^1(z_0)|}\frac{d}{d\tau}|u^1(z_0)|.$$

By $u^1(x_0) \neq 0$, we know that there is a neighborhood $\Gamma \subset \partial M$ of x_0 such that for any $z_0 \in \Gamma$, $u^1(z_0) \neq 0$. Hence (5.3) is equivalent to

$$\frac{d}{d\tau}\left(\frac{|u^0|}{|u^1|}\right)\bigg|_{z_0\in\Gamma} \neq 0, \tag{5.6}$$

which is true thanks to (4.3). Therefore (5.3) holds true, and the lemma is proved. ◻

Lemma 5.2. *Assume* (4.1) *and* (4.2). *If* $\mathrm{ind}(u^0, x_0) =$ *integer, then for any* $\varepsilon > 0$ *small, one of the two vector fields* $u^0 \pm \varepsilon u^1$ *has no singular points on* Γ, *while the other one has at least two singular points on* Γ *with one on each side of* x_0.

Proof. By the Singularity Classification Theorem, Theorem 3.2, the condition that $\mathrm{ind}(u^0, x_0) =$ integer is equivalent to n being even. It is easy to see that one of the two boundary orbits connected to x_0 is stable, and the other boundary orbit is unstable.

For $x_0 \in \partial M$, we take an orthogonal coordinate system (z_1, z_2) with x_0 as the origin, its z_1–axis tangent to ∂M at x_0, and with the z_2–axis pointing inward. In the z–coordinate system, $u^i = (u_1^i(z), u_2^i(z))$, $i = 0, 1$.

Since one of the two boundary orbits connected to x_0 is stable and the other is unstable, we have

$$\mathrm{sign}\, u_1^0(z^-) = \mathrm{sign}\, u_1^0(z^+), \tag{5.7}$$

where $z^\pm = (z_1^\pm, z_2^\pm) \in \Gamma \subset \partial M$ and $z_1^- < 0 < z_1^+$. Without loss of generality, we assume that

$$u_1^0(z) > 0, \quad u_1^1(z) > 0, \quad \text{for } z \in \Gamma \text{ and } z \neq 0. \tag{5.8}$$

Since $u_1^0(0) = 0$, $u_1^1(0) \neq 0$, we infer from (5.8) that for any $\varepsilon > 0$ small, $u^0 + \varepsilon u^1$ has no singular point on Γ.

On the other hand, by

$$\frac{u_2^0(z)}{u_1^0(z)} = \frac{u_2^1(z)}{u_1^1(z)} \quad \forall z \in \Gamma \text{ and } z \neq 0, \tag{5.9}$$

we see that $u^0(z) - \varepsilon u^1(z) = 0$ is equivalent to $u_1^0(z) - \varepsilon u_1^1(z) = 0$. By (5.8) and $u_1^0(0) = 0$, $u_1^1(0) > 0$, we deduce that for any $\varepsilon > 0$ sufficiently small, $u^0 - \varepsilon u^1$ has at least two singular points $z^+(\varepsilon) = (z_1^+, z_2^+)$, $z^-(\varepsilon) = (z_1^-, z_2^-) \in \Gamma$ with $z_1^- < 0 < z_1^+$, such that

$$\lim_{\varepsilon \to 0} z^+(\varepsilon) = 0, \quad \lim_{\varepsilon \to 0} z^-(\varepsilon) = 0.$$

The proof of the lemma is complete. □

Lemma 5.3. *Assume* (4.1) *and* (4.2). *If* $\operatorname{ind}(u^0, x_0) =$ *non-integer, then for any* $\varepsilon > 0$ *small, each* $u^0 \pm \varepsilon u^1$ *has at least one singular point on* Γ.

Proof. When $\operatorname{ind}(u^0, x_0) =$ fraction, $n =$ odd. Hence the two boundary orbits connected to x_0 are either all stable or all unstable.

Let (z_1, z_2) be the z–coordinate system as in the proof of Lemma 5.2. Then

$$\operatorname{sign} u_1^0(z^-) = -\operatorname{sign} u_1^0(z^+), \tag{5.10}$$

where $z^\pm = (z_1^\pm, z_2^\pm) \in \Gamma \subset \partial M$ and $z_1^- < 0 < z_1^+$. We assume that

$$u_1^0(z^-) > 0, \quad u_1^1(0) > 0. \tag{5.11}$$

Then (5.10) and (5.11) yield

$$u_1^0(z^+) < 0, \quad \text{for } z^+ \in \Gamma. \tag{5.12}$$

From (5.9) we see that $u^0(z) \pm \varepsilon u^1(z) = 0$ are equivalent to $u_1^0(z) \pm \varepsilon u_1^1(z) = 0$. By (5.11), (5.12) and $u_1^1(z) > 0$ $\forall z \in \Gamma$, we obtain that for any $\varepsilon > 0$ sufficiently small, there is at least a point $z_0^+ \in \Gamma$ (resp. a point $z_0^- \in \Gamma$) such that

$$u^0(z_0^\pm) \pm \varepsilon u^1(z_0^\pm) = 0. \tag{5.13}$$

□

To complete the proof of Theorem 4.1, it suffices to prove the following lemma.

Lemma 5.4. *Assume* (4.1) *and* (4.2).

(1) *If n = even, one of $u^0 \pm \varepsilon u^1$ has no singular points on Γ and the other has exactly two singular points on $\Gamma \subset \partial M$;*
(2) *If n = odd, $u^0 \pm \varepsilon u^1$ have only one singular point on $\Gamma \subset \partial M$.*

Proof. We prove only the case of $u^0 + \varepsilon u^1$ for n = odd; the other case can be proved in exactly the same manner.

Without loss of generality, we assume (5.11) as in the proof of the previous lemma. Then singularities of $u^0 + \varepsilon u^1$ on Γ can only occur for $z = (z_1, z_2) \in \Gamma$, with $z_1 > 0$. Let $\bar{z}$, $\tilde{z} \in \Gamma$ be two different singular points of $u^0 + \varepsilon u^1$; then

$$\varepsilon = \frac{u_1^0(\bar{z})}{u_1^1(\bar{z})} = \frac{u_1^0(\tilde{z})}{u_1^1(\tilde{z})}, \tag{5.14}$$

which contradicts $u_1^0(z)/u_1^1(z)$ being monotone on each side of x_0 on Γ. Here the monotonicity is due to (4.3). It follows that $u^0 + \varepsilon u^1$ has only one singular point on Γ. Thus both the lemma and, therewith, the theorem are proved. □

5.2. Proof of Theorem 4.2. Let $x_1 \in \overset{\circ}{V}$ be an isolated singular point of $u^0 + \varepsilon u^1$ ($\varepsilon \neq 0$). By the Singular Classification Theorem, there are $2n$ ($n \geq 0$) orbits of $u^0 + \varepsilon u^1$ connected to x_1. We now prove that only the cases $n = 0, 2$ may occur.

Let $v = u^0 + \varepsilon u^1$. If x_1 is a non-degenerate singular point of v, then the conclusion holds true. Therefore we assume that $x_1 \in \overset{\circ}{V}$ is a degenerate singular point of v.

Claim. *If $n \neq 0, 2$, then the Jacobian matrix*

$$Dv(x_1) = 0. \tag{5.15}$$

Proof of Claim (5.15). We divide the proof into six steps as follows.

STEP 1 Let γ be an orbit of v connected to x_1. Let (z_1, z_2) the orthogonal coordinate system with x_1 as its origin, and with its z_1-axis tangent to γ at x_1. Then v can be expressed by

$$v(z) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + o(|z|). \tag{5.16}$$

By definition, the z_1–axis is tangent to γ at x_1 (i.e., $z = 0$), which yields

$$\lim_{\substack{z \in \gamma \\ z \to 0}} \frac{v_2(z)}{v_1(z)} = 0. \tag{5.17}$$

STEP 2 We now prove that $a = 0$ in (5.16). Assuming otherwise, we notice that for $(z_1, z_2) \in \gamma$, $z_2 = o(|z_1|)$. Therefore, we infer from (5.16) and (5.17) that

$$\lim_{\substack{z\in\gamma\\ z\to 0}} \frac{v_2(z)}{v_1(z)} = \lim_{\substack{z\in\gamma\\ z\to 0}} \frac{cz_1 - az_2 + o(|z|)}{az_1 + bz_2 + o(|z|)}$$
$$= \lim_{\substack{z\in\gamma\\ z_1\to 0}} \frac{cz_1 + o(|z_1|)}{az_1 + o(|z_1|)} = \frac{c}{a} = 0.$$

Hence $c = 0$. Since x_1 is a degenerate singular point of v, $c = 0$ implies that $a = 0$, a contradiction.

STEP 3 If $a = 0$, then $bc = 0$ for x_1 is a degenerate singular point. Then we have $c = 0$. Otherwise, $b = 0$, and

$$0 = \lim_{\substack{z\in\gamma\\ z\to 0}} \frac{v_2(z)}{v_1(z)} = \lim_{\substack{z\in\gamma\\ z\to 0}} \frac{cz_1 + o(|z|)}{o(|z|)} \neq 0,$$

a contradiction again. We have thus obtained that

$$v_1(z) = bz_2 + o(|z|), \quad v_2 = o(|z|). \tag{5.18}$$

STEP 4 Consider the case where there is another orbit γ_1 of v connected to x_1, and the angle between γ_1 and γ at x_1 is ϑ different from 0 and π. Then by (5.18) we deduce that

$$\lim_{\substack{z\in\gamma_1\\ z\to 0}} \frac{v_2(z)}{v_1(z)} = \lim_{\substack{z\in\gamma_1\\ z\to 0}} \frac{o(|z|)}{bz_2 + o(|z|)} = \tan\vartheta \neq 0,$$

which yields that $b = 0$. Hence (5.15) holds true in this case.

STEP 5 Consider the case where $n = 1$, and there is another orbit of v connected to x_1 tangent to γ at x_1. By the Singularity Classification Theorem, it is obvious that for any $x_2 > 0$ sufficiently small,

$$\operatorname{sign} v_1(0, x_2) = \operatorname{sign} v_1(0, -x_2),$$

which yields, in combination with (5.18), that $b = 0$.

STEP 6 In the case where $n \geq 2$, and all orbits connected to x_1 are tangent to γ at x_1, let $O \subset M$ be a sufficiently small neighborhood of x_1, F_i $(1 \leq i \leq 2n)$ be the domains in O enclosed by the orbits connected to x_1, and

ϑ_i be the angle of the boundary of F_i at x_1. It is easy to see that in each F_i with $\vartheta_i = 0$, there is at least a curvilinear segment ℓ_i with x_1 being its end point, such that $v_1(z) = 0$, $z \in \ell_i$ (see Figure 5.1 below). Hence, there are at least $2(n-1)$ curvilinear segments in O with x_1 as their common end point where $v_1 = 0$. On the other hand, by the Implicit Function Theorem, if $b \neq 0$ in (5.18), then there is a unique curve $L \subset O$ with $x_1 \in L$ such that $v_1(z) = 0$, $z \in L$, i.e., there are only two line segments $L = \ell_1 \cup \ell_2$ in O along which $v_1 = 0$; hence if $n \neq 0, 2$, it follows that $b = 0$ and (5.15) holds true.

This completes the proof of the claim. □

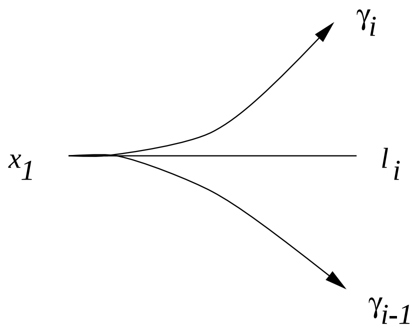

FIGURE 5.1. Auxiliary sketch for Step 6 in proving (5.15).

To complete the proof of Theorem 4.2, it suffices then to prove that $D(v)(x_1) \neq 0$. With the z–coordinate system given in the proof of Lemma 5.2, we have

$$D(v)(x_1) = \begin{pmatrix} \dfrac{\partial u_1^0}{\partial z_1} + \varepsilon \dfrac{\partial u_1^1}{\partial z_1} & \dfrac{\partial u_1^0}{\partial z_2} + \varepsilon \dfrac{\partial u_1^1}{\partial z_2} \\ \dfrac{\partial u_2^0}{\partial z_1} + \varepsilon \dfrac{\partial u_2^1}{\partial z_1} & \dfrac{\partial u_2^0}{\partial z_2} + \varepsilon \dfrac{\partial u_2^1}{\partial z_2} \end{pmatrix}_{z=x_1},$$

where

$$\varepsilon = \frac{-u_1^0(x_1)}{u_1^1(x_1)} = \frac{-u_2^0(x_1)}{u_2^1(x_1)}.$$

If $D(v)(x_1) = 0$, then

$$\frac{\partial}{\partial z_1}\left(\frac{u_i^0}{u_i^1}\right)(x_1) = 0, \quad \frac{\partial}{\partial z_2}\left(\frac{u_i^0}{u_i^1}\right)(x_1) = 0, \quad (i = 1, 2). \tag{5.19}$$

When $\varepsilon \to 0$, the singular point $x_1(\varepsilon) \to x_0$ of $u^0 + \varepsilon u^1$ tends to x_0. Let $\ell \subset M$ be the curvilinear segment expressed by $x_1(\varepsilon)$; from (5.19) we then obtain

$$\text{(5.20)} \qquad \begin{cases} \dfrac{d}{ds}\left(\dfrac{u_i^0}{u_i^1}\right) = 0, \\ \left.\dfrac{u_i^0}{u_i^1}\right|_{s=0} = \left.\dfrac{u_i^0}{u_i^1}\right|_{z=0} = 0, \end{cases} \qquad (i = 1, 2),$$

where s is the arc-length parameter along ℓ, starting from x_0. It follows from (5.20) that $u_i^0(z)/u_i^1(z) = 0$, $z \in \ell$, which means that $u^0(z) = 0$, $z \in \ell$. This contradicts the hypothesis that x_0 is an isolated singular point of u, and the proof is thus complete. ◻

Remark 5.1. In the above proof, if we use the vector fields $v = u^0 + \varepsilon u^1(x, \varepsilon)$, $u^1(x, \varepsilon) = u^1(x) + o(\varepsilon)$, $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$, then (5.20) is replaced by the following equation

$$\text{(5.21)} \qquad \begin{cases} \dfrac{\partial}{\partial s}\left(\dfrac{u_i^0(s)}{u_i^1(s, \varepsilon)}\right) = 0, \quad \text{where } \varepsilon = \varepsilon(s), \text{ and } \varepsilon(0) = 0, \\ \dfrac{u_i^0(0)}{u_i^1(0,0)} = 0. \end{cases}$$

On the other hand, $u^0(0)/u^1(0, \varepsilon) = 0$ for any $\varepsilon \neq 0$ sufficiently small, hence from (5.21) we obtain $u^0(z) = 0$ for $z \in \ell$.

5.3. Proof of Theorem 4.3. We divide the proof into two steps.

STEP 1 (The case where $\operatorname{ind}(u^0, x_0) = $ integer) By Theorem 4.1 on page 167, the number of singular boundary points of $u^0 + \varepsilon u^1$ in a small neighborhood $\Gamma \subset \partial M$ of x_0 is not the same as that for $u^0 - \varepsilon u^1$. Therefore, $u^0 + \varepsilon u^1$ and $u^0 - \varepsilon u^1$ are not topologically equivalent at $x_0 \in \partial M$. Moreover if x_0 is the unique degenerate singular point of u^0 on the boundary ∂M, then $u^0 + \varepsilon u^1$ and $u^0 - \varepsilon u^1$ are not topologically equivalent in a small neighborhood of ∂M. In other words, t_0 is a bifurcation point of the vector field u's global structure.

STEP 2 (The case where $\operatorname{ind}(u^0, x_0) = $ fraction) Let $z_0^+ = (z_1^+, z_2^+) \in \Gamma \subset \partial M$ (resp. $z_0^- = (z_1^-, z_2^-) \in \Gamma$) be the singular point of $u^0 + \varepsilon u^1$ (resp. $u^0 - \varepsilon u^1$) and $z_1^- < 0 < z_1^+$ as in Theorem 4.1 on page 167. Then we have

$$\operatorname{ind}(u^0 \pm \varepsilon u^1, z_0^\pm) = -\frac{1}{2}.$$

By the Singularity Classification Theorem, there is only one orbit $\gamma^+(\varepsilon)$ of $u^0 + \varepsilon u^1$ in $\mathring{M}$ connected to z_0^+ (resp. only one orbit $\gamma^-(\varepsilon)$ of $u^0 - \varepsilon u^1$ in $\mathring{M}$

connected to z_0^-). By the same Theorem 3.2 on page 164, there are n ($n \geq 3$ and odd) orbits γ_i $(1 \leq i \leq n)$ of u^0 in $\overset{\circ}{M}$ connected to $x_0 \in \partial M$ $(z = 0)$; see Figure 5.2 below.

Without loss of generality, we assume that the two orbits near the boundary are stable, and the z_1-component $u_1^1(z)$ of $u^1(z) = \partial u(z, t_0)/\partial t$ satisfies $u_1^1(0) > 0$. Let $V_1 \subset \overset{\circ}{M}$ (resp. $V_2 \subset \overset{\circ}{M}$) be the domain near $x_0 \in \partial M$ enclosed by ∂M and γ_1 (resp. by ∂M and γ_n); see Figure 5.2.

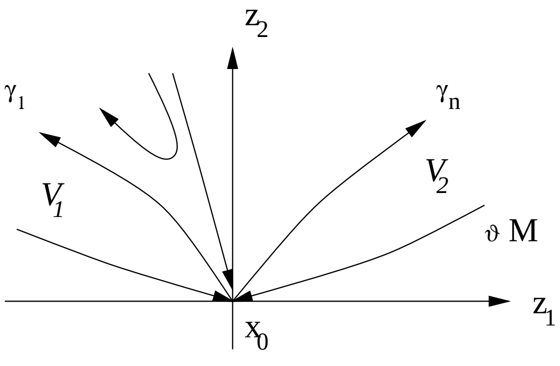

FIGURE 5.2. Auxiliary sketch for Step 2 in the proof of Theorem 4.3.

Since $u_1^1(0) > 0$, the flow of $u^0 + \varepsilon u^1$ crosses γ_n transversally and enters into V_2 (resp. the flow of $u^0 - \varepsilon u^1$ crosses γ_1 transversally and enters V_1). Therefore, we have

$$\gamma^+(\varepsilon) \subset V_2, \quad \gamma^-(\varepsilon) \subset V_1; \tag{5.22}$$

see Figure 5.3. Obviously, we have

$$\lim_{\varepsilon \to 0} \gamma^-(\varepsilon) = \gamma_1, \quad \lim_{\varepsilon \to 0} \gamma^+(\varepsilon) = \gamma_n.$$

Moreover, there are n orbits $\Gamma_i^+ = \Gamma_i^+(\varepsilon)$ of $u^0 + \varepsilon u^1$ in $V - V_1$ (resp. n orbits $\Gamma_i^- = \Gamma_i^-(\varepsilon)$ of $u^0 - \varepsilon u^1$ in $V - V_2$), which are connected to singular points of $u^0 + \varepsilon u^1$ (resp. $u^0 - \varepsilon u^1$) such that

$$\lim_{\varepsilon \to 0} \gamma_i^+(\varepsilon) = \gamma_i, \quad (\text{resp. } \lim_{\varepsilon \to 0} \gamma_i^-(\varepsilon) = \gamma_i),$$

where $V \subset \overset{\circ}{M}$ is a neighborhood of x_0.

Let U_1^+ (resp. U_1^-) be the domain enclosed by ∂M and γ^+ (resp. by ∂M and γ^-), and $U_2^+ = V - U_1^+$ (resp. $U_2^- = V - U_1^-$). We infer from (5.22) that

$$\gamma_i^+ \subset U_1^+, \quad \gamma_i^- \subset U_2^- \quad (1 \leq i \leq n). \tag{5.23}$$

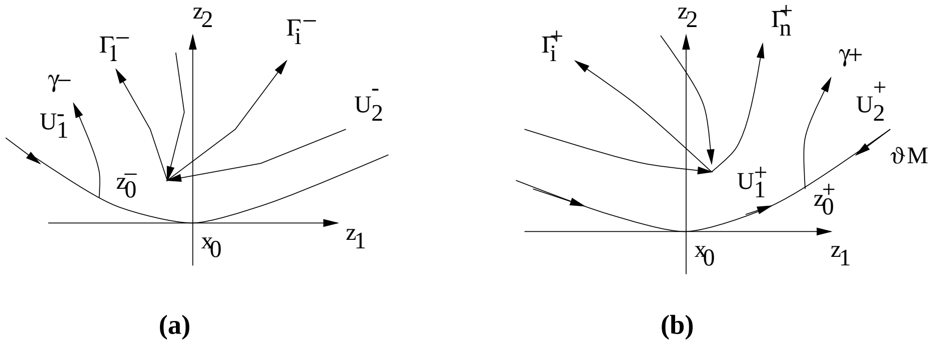

FIGURE 5.3. Auxiliary sketch for the final step in the proof of Theorem 4.3.

We know that under topological equivalence, the orbits of $u^0 + \varepsilon u^1$ connected to a singular boundary point are mapped to the orbits of $u^0 - \varepsilon u^1$ connected to a boundary singular point, preserving orientations. Hence if $u^0 + \varepsilon u^1$ and $u^0 - \varepsilon u^1$ are topologically equivalent locally at $x_0 \in \partial M$, then the restriction of $u^0 + \varepsilon u^1$ to U_1^+ has to be topologically equivalent to the restriction to $u^0 - \varepsilon u^1$ in U_1^-. This is in contradiction with (5.23), and Assertion 1 of Theorem 4.3 on page 168 thus holds true.

Assertion 2 can be proved in the same fashion, completing therewith the proof of Theorem 4.3 on page 168.

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REFERENCES

[1] V. ARNOLD, *Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid*, Dokl. Akad Nauk SSSR **162** (1965), 975-978.

[2] V. ARNOLD & B. KHESIN, *Topological Methods in Hydrodynamics*, Springer-Verlag, New York, 1998.

[3] G.I. BATCHELOR, *An Introduction to Fluid Mechanics*, Cambridge University Press, London, 1967.

[4] YANN BRENIER, *The least action principle and the related concept of generalized flows for incompressible perfect fluids*, J. Amer. Math. Soc. **2** (1989), 225-255.

[5] ______, *Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations*, Comm. Pure Appl. Math. **52** (1999), 411-452.

[6] L. CAFFARELLI, R. KOHN & L. NIRENBERG, *On the regularity of the solutions of Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982), 771-831.
[7] P. CONSTANTIN & C. FOIAS, *The Navier-Stokes Equations*, Univ. of Chicago Press, Chicago, 1988.
[8] M. GHIL & S. CHILDRESS, *Topics in Geophysical Fluid Dynamics: Atmospheric Dynamics, Dynamo Theory, and Climate Dynamics*, Springer-Verlag, New York, 1987.
[9] D.B. HAIDVOGEL, J.C. MCWILLIAMS & P.R. GENT, *Boundary current separation in a quasi-geostrophic, eddy-resolving ocean circulation model*, J. Phys. Oceanogr. **11** (1992), 557-568.
[10] S. JIANG, F.-F. JIN & M. GHIL, *Multiple equilibria, periodic and aperiodic solutions in a wind-driven, double-gyre, shallow-water model*, J. Phys. Oceanogr. **25** (1995), 764-786.
[11] J.L. LIONS, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.
[12] J.L. LIONS, R. TEMAM & S. WANG, *Geostrophic asymptotics of the primitive equations of the atmosphere*, Topological Methods in Nonlinear Analysis **4** (1994), special issue dedicated to J. Leray, 253-287.
[13] ———, *A simple global model for the general circulation of the atmosphere*, (Dedicated to Peter D. Lax and Louis Nirenberg on the occasion of their 70th birthdays), Comm. Pure. Appl. Math. **50** (1997), 707-752.
[14] T. MA & S. WANG, *Structural classification and stability of divergence-free vector fields*, submitted, 1998.
[15] ———, *Dynamics of incompressible vector fields*, Applied Math. Lett. **12** (1999), 39-42.
[16] ———, *The geometry of the stream lines of steady states of the Navier-Stokes equations*, Contemporary Mathematics, AMS **238** (1999), 193-202.
[17] A. MAJDA, *Vorticity and the mathematical theory of incompressible fluid flow. Frontiers of the Mathematical Sciences*, New York, 1985, Comm. Pure Appl. Math. **39** (1986), S187-S220.
[18] J. PALIS & W. DE MELO, *Geometric Theory of Dynamical Systems*, Springer-Verlag, New York, 1982.
[19] J. PEDLOSKY, *Geophysical Fluid Dynamics*, 2nd Edition, Springer-Verlag, New York, 1987.
[20] M. PEIXOTO, *Structural stability on two-dimensional manifolds*, Topology **1** (1962), 101-120.
[21] CHARLES C. PUGH, *The closing lemma*, Amer. J. Math. **89** (1967), 956-1009.
[22] C. ROBINSON, Generic properties of conservative systems, I, II, Amer. J. Math. **92** (1970), 562-603 and 897-906.
[23] ———, *Structural stability of vector fields*, Ann. of Math. **99** (1974), 154-175.
[24] MICHAEL SHUB, *Stabilité globale des systèmes dynamiques*, Astérisque **56** (1978) Société Mathématique de France, Paris. With an English preface and summary.
[25] S. SMALE, *Differential dynamical systems*, Bull. AMS **73** (1967), 747-817.
[26] R. TEMAM, *Navier-Stokes Equations, Theory and Numerical Analysis*, 3rd, rev. ed., North Holland, Amsterdam, 1984.
[27] C. TRUESDELL, *The Kinematics of Vorticity*, Indiana University Press, Bloomington, 1954.

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