SIAM J. APPL. MATH.
Vol. 0, No. 0, pp. 000–000

STRUCTURAL BIFURCATION OF 2-D NONDIVERGENT FLOWS WITH DIRICHLET BOUNDARY CONDITIONS: APPLICATIONS TO BOUNDARY-LAYER SEPARATION*

MICHAEL GHIL†, TIAN MA‡, AND SHOUHONG WANG§

Abstract. This article addresses transitions in the topological structure of a family of divergence-free vector fields $u(\cdot, t)$ with Dirichlet boundary conditions. We show that structural bifurcation—i.e., change in topological-equivalence class—occurs at t_0 if $u(\cdot, t_0)$ has an isolated degenerate ∂-singular point $\bar{x} \in \partial M$ such that $\partial^2 u(\bar{x}, t_0)/\partial n \partial t \neq 0$. The main results are proved by classifying orbit structures of u near such a point $\bar{x} \in \partial M$ of $u(\cdot, t_0)$. The condition of $\bar{x}$ being a ∂-singular point is equivalent to the one originally postulated by Prandtl for boundary-layer separation. Our analysis and classification do contribute, in fact, to a rigorous characterization of boundary-layer separation in 2-D incompressible fluid flows.

Key words. structural bifurcation, boundary layer separation, 2-D incompressible flows

AMS subject classifications. 37G, 76M, 34D30, 35Q30, 37E

DOI. 10.1137/S0036139903438818

1. Introduction.

This article is part of a research program on the use of topological ideas to study the spatio-temporal structure of 2-D incompressible fluid flows in physical space, along with its stability and bifurcations. This program consists of research in two areas: (a) the study of the topological structure of divergence-free vector fields, and its evolution in time or with respect to an arbitrary parameter; and (b) the study of the structure and evolution of velocity fields for 2-D incompressible fluid flows governed by a class of equations that comprises the Navier–Stokes equations, the Euler equations, and the quasi-geostrophic equations of rotating flows.

Mathematically speaking, there are two general methods for describing a fluid flow: the Euler representation and the Lagrange representation; see [1, 2, 3, 6, 9, 13, 17]. In the Euler representation, the motion of a fluid is described by a set of partial differential equations (PDEs)—such as the Euler equations or the Navier–Stokes equations, supplemented with proper boundary conditions—that govern the velocity field at every point in the (2-D or 3-D) flow domain. The Lagrange representation of a fluid flow, on the other hand, amounts to studying the trajectories of fluid particles

*Received by the editors December 15, 2003; accepted for publication (in revised form) October 24, 2004; published electronically DATE.
http://www.siam.org/journals/siap/x-x/43881.html

†Département Terre-Atmosphère-Océan, Ecole Normale Supérieure, F-75231 Paris, Cedex 05, France, and Institute of Geophysics and Planetary Physics, University of California, Los Angeles, CA 90095. This author was supported in part by the National Science Foundation under grants from the Atmospheric Sciences and the Oceanic Sciences Divisions.

‡Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, People's Republic of China, and Department of Mathematics, Indiana University, Bloomington, IN 47405. The research of this author was supported in part by the Office of Naval Research under a grant from the Mathematical, Computer, and Information Sciences Division, by the National Science Foundation under a grant from the Division of Mathematical Sciences (DMS), and by the National Science Foundation of China.

§Department of Mathematics, Indiana University, Bloomington, IN 47405 (showang@indiana.edu, http://www.indiana.edu/~fluid). The research of this author was supported in part by the Office of Naval Research under a grant from the Mathematical, Computer, and Information Sciences Division, by the National Science Foundation under a grant from the Division of Mathematical Sciences (DMS), and by the National Science Foundation of China.

as a function of initial position in the flow domain, subject to the ordinary differential equations (ODEs) that govern the change in position given the velocity. Of course the velocities of the particles satisfy the PDEs we just mentioned.

Our approach is to classify the topological structure of the *instantaneous* velocity field, treating the time variable as a parameter, and the changes in this structure with respect to time. The aforementioned two areas of our program draw inspiration from and are relevant to both the Eulerian and the Lagrangian approaches to fluid flows.

The study in area (a) is kinematic in nature, and the results and methods developed can naturally be applied to other problems of mathematical physics that involve divergence-free vector fields. These include, for instance, problems in electromagnetism in which the magnetic field is necessarily divergence-free. The main topics in this area include structural classification, structural stability, and structural bifurcation, as well as their applications to fluid dynamics in general and to geophysical fluid dynamics in particular. The study in area (b) involves specific connections between the solutions of the evolution equations—whether Navier–Stokes, Euler, or quasi-geostrophic—and flow structure in the physical space.

The main objective of this paper is to contribute to a rigorous characterization of boundary-layer separation in 2-D incompressible fluid flows. This is a long-standing problem in fluid mechanics that goes back to the pioneering work of Prandtl [15] in 1904. Classical boundary-layer theory is presented in [2, 9, 16]. The Prandtl equation represents an approximation of the Navier–Stokes equations inside the boundary layer in the absence of separation; this equation is rigorously analyzed in a recent textbook [14] and several articles [4, 5, 18].

Basically, the boundary layer is a narrow region of sharp velocity gradients between a no-slip wall, where the velocity has to vanish, and the interior of the fluid. This layer of high shear can detach from the boundary, generating vortices and leading to more complicated turbulent behavior near the wall as well as in the interior of flow domain [9]. It is important, therefore, to characterize, if at all possible, the conditions for separation. Experimentally one observes that the normal derivative of the velocity field vanishes at or near separation points. Chorin and Marsden [2] note that there is no known theorem that can be applied to determine the separation reliably. This article, along with [8, 10], is an attempt to derive a rigorous characterization of streamline detachment from the boundary for 2-D divergence-free vector fields. These results are applied in the companion papers [7, 12] to the actual problem of boundary-layer separation in solutions of the 2-D incompressible Navier–Stokes equations.

In the present article, we address structural transitions for a family of divergence-free vector fields $u(\cdot, t)$ with Dirichlet boundary conditions. We show that structural bifurcation—i.e., change in their topological-equivalence class—occurs at t_0 if $u(\cdot, t_0)$ has an isolated degenerate ∂-singular point $\bar{x} \in \partial M$, such that $\partial^2 u(\bar{x}, t_0)/\partial n \partial t \neq 0$. The condition that $\bar{x} \in \partial M$ is a ∂-singular point is related, as we shall see in section 3, to the Prandtl condition [15] for boundary-layer separation.

Our main results are based on a complete classification of orbit structures near an isolated degenerate ∂-singular point. These results extend over several papers and rely on a delicate analysis of the flow structure near the boundary for both free-slip and Dirichlet boundary conditions. The first step was to classify the flow structure and its transitions near the boundary for flows subject only to boundary conditions of zero normal flow, often called free-slip conditions in fluid dynamics [8]. Second, Ma and Wang [10] analyzed the case of Dirichlet boundary conditions for a 2-D divergence-free vector field; in fluid mechanics the Dirichlet condition on the velocity is often called the no-slip condition.

Technically speaking, homogeneous Dirichlet boundary conditions for $u(\cdot, t_0)$ imply that all points on ∂M are singular points in the usual sense. Hence, to analyze and to classify the structure of u near the boundary, including the separation point, we need to use the concept of a ∂-singular point, introduced in [10], which corresponds to singular points of the normal derivative of u in the usual sense. Finally, in the present paper we make the connection between the structure of the original velocity fields and the structure of the normal derivative of the velocity field.

The paper is organized as follows. In section 2, we summarize our previous results, including a structural stability theorem, necessary conditions on structural bifurcation, and a singularity classification theory for 2-D divergence-free vector fields. Section 3 states the structural-bifurcation theorems near a flat boundary for such fields, which are proved in section 4. Section 5 addresses structural bifurcations near a curved boundary, and section 6 applies the theory to streamline detachment from the boundary. It is the results of section 6 that are applied in the companion papers [7, 12] to boundary-layer separation in the 2-D Navier–Stokes equations for incompressible flows.

2. Preliminaries. Let $M \subset \mathbb{R}^2$ be a closed and bounded domain with C^{r+1} $(r \geq 2)$ boundary ∂M, and let TM be the tangent bundle of M. Let $C^r_n(TM)$ be the space of all C^r vector fields on M that satisfy a boundary condition of no normal flow (or no penetration), let $D^r(TM)$ be the subspace of $C^r_n(TM)$ that is divergence-free, and let $B^r_0(TM)$ be the subspace of $D^r(TM)$ that satisfies a homogeneous Dirichlet boundary condition:

$$
\begin{aligned}
&C^r_n(TM) = \{u \in C^r(TM) \mid u_n|_{\partial M} = 0\},\\
&D^r(TM) = \{u \in C^r(TM) \mid u_n|_{\partial M} = 0, \text{ div} u = 0\},\\
&B^r_0(TM) = \{u \in D^r(TM) \mid u|_{\partial M} = 0\}.
\end{aligned}
$$

Here $u_n = u \cdot n$ and $u_\tau = u \cdot \tau$, while n and τ are the unit normal and tangent vector on ∂M, respectively. It is easy to see that

$$
B^r_0(TM) \subset D^r(TM) \subset C^r_n(TM) \subset C^r(TM).
$$

We start with some basic concepts. Let $X = D^r(TM)$ or $B^r_0(TM)$ in the following four definitions.

Definition 2.1. *Two vector fields $u, v \in X$ are called topologically equivalent in X if there exists a homeomorphism of $\varphi : M \to M$, which takes the orbits of u to the orbits of v and preserves their orientation.*

Definition 2.2. *Let $u \in C^1([0, T], X)$ be a one-parameter family of vector fields in X. The vector field $u_0 = u(\cdot, t_0)$ $(0 < t_0 < T)$ is called a bifurcation point of u at time t_0 if, for any $t^- < t_0$ and $t_0 < t^+$ with t^- and t^+ sufficiently close to t_0, the vector field $u(\cdot, t^-)$ is not topologically equivalent to $u(\cdot, t^+)$. In this case, we say that $u(x, t)$ has a bifurcation at t_0 in its global structure.*

Definition 2.3. *Let $u \in C^1([0, T], X)$. We say that $u(x, t)$ has a bifurcation in its local structure in a neighborhood $U \subset M$ of x_0 at t_0 $(0 < t_0 < T)$ if, for any $t^- < t_0$ and $t_0 < t^+$ with t^- and t^+ sufficiently close to t_0, the vector fields $u(\cdot, t^-)$ and $u(\cdot, t^+)$ are not topologically equivalent locally in $U \subset M$.*

We remark here that bifurcation in the vector field's local structure does not imply bifurcation in its global structure. In fact, one can easily construct examples

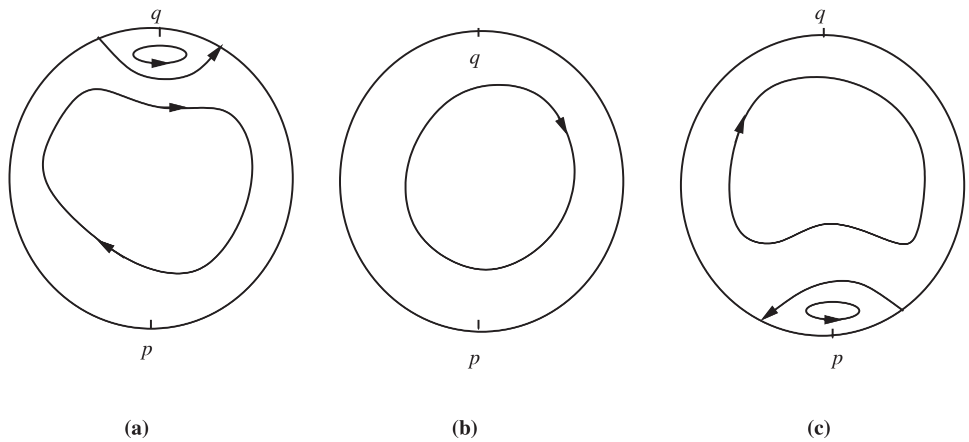

FIG. 2.1. *Flow structure* (a) *for* $t = t^- < t_0$, (b) *for* $t = t_0$, *and* (c) *for* $t = t^+ > t_0$. *Bifurcations in local structure occur at* $t = t_0$ *near both* p *and* q, *but there is no bifurcation in global structure when going from* (a) *to* (c).

showing that flow structure changes in some local area $U \subset M$, but not on the whole manifold M; see, for instance, the flow transitions shown in Figure 2.1.

DEFINITION 2.4. *A vector field* $v \in X$ *is called structurally stable in* X *if there exists a neighborhood* $\mathcal{O} \subset X$ *of* v *such that for any* $u \in \mathcal{O}$, u *and* v *are topologically equivalent.*

A point $p \in M$ is called a singular point of $u \in C^r_n(TM)$ if $u(p) = 0$; a singular point p of u is called nondegenerate if the Jacobian matrix $Du(p)$ is invertible; u is called regular if all singular points are nondegenerate.

By definition, in the case where homogeneous Dirichlet boundary conditions are satisfied, all points on the boundary are singular points. In order to classify the structure of the divergence-free vector fields near the boundary in this case, and infer the possible bifurcations in this structure, we need to distinguish between different types of singular points on the boundary. The concepts of ∂-regular and ∂-singular points introduced in [10] are crucial in order to study the topological structure of divergence-free vector fields with homogeneous Dirichlet boundary conditions.

Let $p \in \partial M$, and let U be a neighborhood of p. Then on $\partial M \cap U$ there exist unit tangent and normal vector fields τ and n. For U sufficiently small, we can extend these two vector fields to U so that the orbits of n in U are tangent to λn with n restricted to $\partial M \cap U$; here $0 \leq \lambda \leq 1$. Note that when U is sufficiently small, for any two points $x, y \in \partial M \cap U$, λn_x and λn_y do not intersect within U. The extension of τ in U is taken to be orthogonal to n.

DEFINITION 2.5. *Let* $u \in B^r_0(TM)$ $(r \geq 2)$.

(i) *A point* $p \in \partial M$ *is called a* ∂*-regular point of* u *if* $\partial u_\tau(p)/\partial n \neq 0$; *otherwise,* $p \in \partial M$ *is called a* ∂*-singular point of* u.

(ii) *A* ∂*-singular point* $p \in \partial M$ *of* u *is called nondegenerate if*

$$\det \begin{pmatrix} \dfrac{\partial^2 u_\tau(p)}{\partial \tau \partial n} & \dfrac{\partial^2 u_\tau(p)}{\partial n^2} \\[2ex] \dfrac{\partial^2 u_n(p)}{\partial \tau \partial n} & \dfrac{\partial^2 u_n(p)}{\partial n^2} \end{pmatrix} \neq 0.$$

A nondegenerate ∂-singular point of u is also called a ∂-saddle point of u.

DEFINITION 2.6. *$u \in B_0^r(TM)$ $(r \geq 2)$ is called D-regular if*

(i) *u is regular in $\overset{\circ}{M}$, the interior of M, and*
(ii) *all ∂-singular points of u on ∂M are nondegenerate.*

For a D-regular divergence-free vector field v on M, an interior nondegenerate singular point of v can be either a center or a saddle, and saddles of v must be connected to saddles. An interior saddle $p \in \overset{\circ}{M}$ is called *self-connected* if p is connected only to itself, i.e., p occurs in a graph whose topological form is that of the number 8.

The following structural stability theorem in the presence of homogeneous Dirichlet boundary conditions was proved in [10].

THEOREM 2.7 (Ma and Wang [10]). *Let $u \in B_0^r(TM)$ $(r \geq 2)$. Then u is structurally stable in $B_0^r(TM)$ if and only if*

1. *u is D-regular;*
2. *all interior saddle points of u are self-connected; and*
3. *each ∂-saddle point of u on ∂M is connected to a ∂-saddle point on the same connected component of ∂M.*

Moreover, the set of all structurally stable vector fields is open and dense in $B_0^r(TM)$.

Based on Theorem 2.7, the following theorem gives some necessary conditions for structural bifurcation.

THEOREM 2.8. *Let $u \in C^1([0,T], B_0^r(TM))$ $(r \geq 2)$.*

1. *If $u(x,t)$ has a bifurcation in its local structure in an arbitrarily small neighborhood $U \subset M$ of x_0 at t_0 $(0 < t_0 < T)$, then $x_0 \in \overset{\circ}{M}$ (respectively, $x_0 \in \partial M$) must be a degenerate singular point (respectively, degenerate ∂-singular point) of $u(x,t)$ at t_0.*
2. *If $u(x,t)$ has a bifurcation in its global structure at t_0 $(0 < t_0 < T)$, then $u(x,t_0)$ does not satisfy at least one of the conditions* (1)–(3) *in Theorem* 2.7.

To proceed, we need to recall the definition of indices of singular points of a vector field. Let $p \in M$ be an isolated singular point of $v \in C_n^r(TM)$; then

$$\text{ind}(v,p) = \deg(v,p),$$

where $\deg(v,p)$ is the Brouwer degree of v at p.

Let $p \in \partial M$ be an isolated singular point of v, and let $\widetilde{M} \subset \mathbb{R}^2$ be an extension of M, i.e., $M \subset \widetilde{M}$ such that $p \in \widetilde{M}$ is an interior point of $\widetilde{M}$. In a neighborhood of p in $\widetilde{M}$, v can be extended by reflection to $\tilde{v}$ such that p is an interior singular point of $\tilde{v}$, thanks to the no normal flow condition, i.e., $v \cdot n|_{\partial M} = 0$. Then we define the index of v at $p \in \partial M$ by

$$\text{ind}(v,p) = \frac{1}{2}\,\text{ind}(\tilde{v},p).$$

Let $p \in M$ be an isolated singular point of $v \in C_n^r(TM)$. An orbit γ of v is said to be a stable orbit (respectively, an unstable orbit) connected to p if the limit set $\omega(x) = p$ (respectively, $\alpha(x) = p$) for any $x \in \gamma$.

We now introduce a singularity classification theorem for incompressible vector fields, which will be useful in our discussion of structural bifurcation.

THEOREM 2.9 (Ghil, Ma, and Wang [8]). *Let $p \in M$ be an isolated singular point of $v \in D^r(TM)$, $r \geq 1$. Then p is connected only to a finite number of orbits, and the stable and unstable orbits connected to p alternate when tracing a closed curve around p. Furthermore*

1. *when* $p \in \overset{\circ}{M}$, *p has* $2n$ $(n \geq 0)$ *orbits, n of which are stable, and the other n unstable, while the index of p is*

$$\mathrm{ind}(v, p) = 1 - n;$$

2. *when* $p \in \partial M$, *p has* $n + 2$ $(n \geq 2)$ *orbits, two of which are on the boundary* ∂M, *and the index of p is*

$$\mathrm{ind}(v, p) = -\frac{n}{2}.$$

No confusion should arise between the integer n, used for counting orbits, and the notation n for the normal direction to the boundary ∂M.

3. Structural bifurcations near a flat boundary. In this section, we assume that the boundary ∂M contains a flat part $\Gamma \subset \partial M$, and consider structural bifurcation near a ∂-singular point $\bar{x} \in \Gamma$. For simplicity, we take a coordinate system (x_1, x_2) with $\bar{x}$ at the origin and with Γ given by

$$\Gamma = \{(x_1, 0) \,\big|\, |x_1| \leq \delta\}$$

for some $\delta > 0$. Obviously, the tangent and normal vectors on Γ are the unit vectors in the x_1- and x_2-directions, respectively.

Let $u \in C^1([0, T], B_0^r(TM))$ $(r \geq 2)$ be a one-parameter family of divergence-free vector fields subject to homogeneous Dirichlet boundary conditions. In a neighborhood $U \subset M$ of $\bar{x} \in \Gamma$, $u(x, t)$ can be expressed near $x = 0$ by

$$u(x, t) = x_2 v(x, t). \tag{3.1}$$

It is easy to see that u and v have topologically equivalent streamlines in an interior neighborhood of $x = 0$. To proceed, we consider the Taylor expansions of both $u(x, t)$ and $v(x, t)$ at t_0 $(0 < t_0 < T)$:

$$\begin{cases} u(x,t) = u^0(x) + (t - t_0)u^1(x) + o(|t - t_0|^2), \\ u^0(x) = u(x, t_0), \\ u^1(x) = \dfrac{\partial u(x, t_0)}{\partial t}, \end{cases} \tag{3.2}$$

$$\begin{cases} v(x,t) = v^0(x) + (t - t_0)v^1(x) + o(|t - t_0|)^2, \\ v^0(x) = v(x, t_0), \\ v^1(x) = \dfrac{\partial v(x, t_0)}{\partial t}. \end{cases} \tag{3.3}$$

Let $u^i = (u_1^i, u_2^i)$, $v^i = (v_1^i, v_2^i)$, $i = 0, 1$. We start with the following conditions for structural bifurcation.

Assumption (H). Let $\bar{x} = 0 \in \Gamma$ be an isolated degenerate ∂-singular point of $u^0(x)$, $u^0 \in C^{k+1}$ near $\bar{x} \in \Gamma$ for some $k \geq 2$. Assume that

$$\frac{\partial u^0(0)}{\partial n} = 0, \tag{3.4}$$

$$\mathrm{ind}(v^0, 0) \neq -\frac{1}{2}, \tag{3.5}$$

$$\frac{\partial u^1(0)}{\partial n} \neq 0, \tag{3.6}$$

$$\frac{\partial^{k+1} u_1^0(0)}{\partial^k \tau \partial n} \neq 0. \tag{3.7}$$

Some remarks are now in order.

Remark 3.1. Condition (3.4) says that $\bar{x} = 0 \in \Gamma$ is a ∂-singular point of $u^0(x)$. In a 2-D incompressible flow, governed by either the Euler or the Navier–Stokes equations, this condition is equivalent to the leading-order vorticity vanishing at $\bar{x}$. The latter is the so-called Prandtl condition, which Prandtl suggested might identify boundary-layer separation points in incompressible flows [15].

Remark 3.2. Condition (3.5) amounts to saying that there exists a number $n \neq 1$ of interior orbits of v^0 connected to $\bar{x} \in \Gamma$. Since $u^0 = x_2 v^0$, the number of interior orbits of u^0 connected to $\bar{x} \in \Gamma$ is exactly $n \neq 1$ as well. This shows that $\bar{x} \in \Gamma$ is a degenerate ∂-singular point of $u^0(x)$, which is necessary for structural bifurcation, according to our structural stability and bifurcation theorems; see Theorems 2.7 and 2.8 here or [8, 10].

Remark 3.3. Condition (3.6) states that the first-order term u^1 of the Taylor expansion for the normal derivative of u is different from zero. This is just the simplest necessary condition of such a type; if it does not hold, we need to work on a higher-order Taylor expansion, and the corresponding results proved in this article will be true as well. In fluid-mechanics applications, condition (3.6) is equivalent to the vorticity associated with u^1 not vanishing at the boundary. In addition, it is easy to see that (3.6) is equivalent to

$$\frac{\partial u_1^1(0)}{\partial x_2} = \frac{\partial u_1^1(0)}{\partial n} \neq 0,$$

which shows that the acceleration of the fluid in the tangential direction near $\bar{x}$ is nonzero.

Remark 3.4. Condition (3.7) is a technical condition and amounts to saying that the tangential component u_1^0 of the leading-order term has a nontrivial Taylor expansion. Furthermore, let k be the smallest integer satisfying condition (3.7). It is easy to show that $k \geq 2$. In fact, $u^0(x)$ has the Taylor expansion at $x = 0$,

$$u^0(x) = \begin{cases} cx_2 + 2ax_1x_2 + bx_2^2 + x_2h_1(x), \\ \qquad\qquad\quad - ax_2^2 + x_2h_2(x), \end{cases} \tag{3.8}$$

with $h_i(x) = o(|x|)$ $(i = 1, 2)$. Since $\bar{x} \in \Gamma$ is a degenerate ∂-singular point of $u^0(x)$, it follows that $c = 0$, $a = 0$, which implies that $k \geq 2$.

The structural bifurcation of $u(x, t)$ near a degenerate ∂-singular point on a flat boundary segment is described by the following theorems.

THEOREM 3.5. *Let $u \in C^1([0, T], B_0^r(TM))$ $(r \geq 2)$ satisfy Assumption* (H). *Then there exist a neighborhood*

$$\Gamma_0 = \{(x_1, 0) \,\big|\, |x_1| \leq \delta_0\} \subset \Gamma$$

of $\bar{x} = 0$ and an $\varepsilon_0 > 0$ such that all ∂-singular points of $u(\cdot, t_0 \pm \varepsilon)$ are nondegenerate for any $0 < \varepsilon \leq \varepsilon_0$. Moreover,

1. *if the index* $\mathrm{ind}(v^0, 0)$ *is an integer, then one of $u(x, t_0 \pm \varepsilon)$ has exactly two ∂-singular points on Γ_0, and the other has no ∂-singular points on Γ_0; and*
2. *if the index* $\mathrm{ind}(v^0, 0)$ *is not an integer, then each of $u(x, t_0 \pm \varepsilon)$ has exactly one ∂-singular point on Γ_0.*

THEOREM 3.6 (structural bifurcation theorem). *Let $u \in C^1([0, T], B_0^r(TM))$ $(r \geq 2)$ satisfy Assumption* (H). *Then*

1. *the vector field u has a bifurcation in its local structure at $(\bar{x}, t_0)$; and*

2. *if $\bar{x} \in \partial M$ is a unique ∂-singular point of u with the same index as* $\mathrm{ind}(v^0, 0)$ *on ∂M, then $u(x,t)$ has a bifurcation in its global structure at $t = t_0$.*

The proofs of these two theorems are based on analyzing the orbits of u to provide a complete classification of the local structure of u near $(\bar{x}, t_0)$. The bifurcation results follow immediately from the classification.

4. Proof of the main theorems.

4.1. Proof of Theorem 3.5. The proof of the theorem is a direct consequence of the following three propositions.

PROPOSITION 4.1. *There exist a neighborhood $\Gamma_0 \subset \Gamma$ of $\bar{x} = 0$ and an $\varepsilon_0 > 0$ such that all ∂-singular points of $u(\cdot, t_0 \pm \varepsilon)$ are nondegenerate for any $0 < \varepsilon \leq \varepsilon_0$.*

Proof. Let $x^* \in \Gamma$, $x^* = (x_1^*)$, and $0 < |x_1^*| \leq \delta_0$ with $0 < \delta_0$ sufficiently small, to be chosen later. Then

$$\frac{\partial u}{\partial n}(x^*, t_0 \pm \varepsilon) = \frac{\partial u}{\partial x_2}(x^*, t_0 \pm \varepsilon) = 0. \tag{4.1}$$

By the Taylor expansion (3.2) and condition (3.6) it suffices to consider only the first-order approximation of (3.2). Hence,

$$\frac{\partial u_i^0(x_1^*, 0)}{\partial x_2} \pm \varepsilon \frac{\partial u_i^1(x_1^*, 0)}{\partial x_2} = 0 \qquad (i = 1, 2). \tag{4.2}$$

We need to show that

$$\det \begin{pmatrix} \dfrac{\partial^2 u_1^0}{\partial x_1 \partial x_2} \pm \varepsilon \dfrac{\partial^2 u_1^1}{\partial x_1 \partial x_2} & \dfrac{\partial^2 u_1^0}{\partial x_2^2} \pm \varepsilon \dfrac{\partial^2 u_1^1}{\partial x_2^2} \\ \dfrac{\partial^2 u_2^0}{\partial x_1 \partial x_2} \pm \varepsilon \dfrac{\partial^2 u_2^1}{\partial x_1 \partial x_2} & \dfrac{\partial^2 u_2^0}{\partial x_2^2} \pm \varepsilon \dfrac{\partial^2 u_2^1}{\partial x_2^2} \end{pmatrix}_{x=(x_1^*, 0)} \neq 0. \tag{4.3}$$

For $u \in C^1([0,T], B_0^r(TM))$, $u(x_1, 0, t) = 0$ for all $(x_1, 0) \in \Gamma$ and $0 \leq t \leq T$. Thus we obtain

$$\frac{\partial u_1}{\partial x_1}(x_1, 0, t) = 0 \qquad \forall\, |x_1| \leq \bar{\delta}.$$

Thanks to the fact that u is divergence-free, we have

$$\frac{\partial u_2}{\partial x_2}(x_1, 0, t) = -\frac{\partial u_1}{\partial x_1}(x, 0, t) = 0 \qquad \forall\, |x_1| \leq \bar{\delta}.$$

Consequently

$$\frac{\partial u_2^0(x_1, 0)}{\partial x_2} \pm \varepsilon \frac{\partial u_2^1(x_1, 0)}{\partial x_2} = 0 \qquad \forall\, |x_1| \leq \bar{\delta},$$

which yields

$$\frac{\partial^2 u_2^0(x_1^*, 0)}{\partial x_1 \partial x_2} \pm \varepsilon \frac{\partial^2 u_2^1(x_1^*, 0)}{\partial x_1 \partial x_2} = 0.$$

To verify (4.3), it suffices to prove that

$$\frac{\partial^2 u_1^0(x_1^*, 0)}{\partial x_1 \partial x_2} \pm \varepsilon \frac{\partial^2 u_1^1(x_1^*, 0)}{\partial x_1 \partial x_2} \neq 0,$$

since the sum of the diagonal terms in (4.3) is zero thanks to the fact that $\partial u/\partial x_2$ is divergence-free.

From conditions (3.7) and (3.6) we get

$$\begin{cases} \dfrac{\partial u_1^0(x_1,0)}{\partial x_2} = \alpha x_1^k + o(|x_1|^k), \quad \alpha \neq 0, \\ \dfrac{\partial u_1^1(x_1,0)}{\partial x_2} = \beta + o(|x_1|), \quad \beta \neq 0. \end{cases} \tag{4.4}$$

Therefore it follows from (4.2) that

$$\varepsilon = \pm \frac{\alpha}{\beta} x_1^{*^k} + o(|x_1^*|^k). \tag{4.5}$$

Thus from (4.4) and (4.5) we obtain that

$$\frac{\partial^2 u_1^0(x_1^*,0)}{\partial x_1 \partial x_2} \pm \varepsilon \frac{\partial^2 u_1^1(x_1^*,0)}{\partial x_1 \partial x_2} = \alpha k x_1^{*^{k-1}} + o\left(|x_1^*|^{k-1}\right),$$

which is different from zero for $0 < |x_1^*| \leq \delta_0$, provided that δ_0 is sufficiently small. Hence (4.3) follows, and the proof of the proposition is complete. □

PROPOSITION 4.2. *If* $\text{ind}(v^0, 0) =$ *integer, then one of* $u(x, t_0 \pm \varepsilon)$ *has no* ∂*-singular points on* Γ_0*, and the other one has exactly two* ∂*-singular points on* Γ_0 *with one* ∂*-singular point on each side of* $\bar{x} = 0$.

Proof. From (3.1) it is easy to see that the zero points of $v(x,t)$ on Γ are equivalent to the ∂-singular points of $u(x,t)$. Hence we only have to prove the assertion for the vector field $v(x, t_0 \pm \varepsilon)$.

According to (3.8) we have, for the component v_2 that is normal to Γ,

$$v_2(x_1, 0, t) = 0 \quad \forall\, x_1 \in \Gamma,\, t \geq 0.$$

By (3.6) and (3.7) we infer from (3.1) that

$$\begin{cases} v_1^0(x_1,0) = \alpha x_1^k + o(|x_1|^k), & \alpha \neq 0, \\ v_1^1(x_1,0) = \beta + g(x_1), & \beta \neq 0,\, g(0) = 0. \end{cases} \tag{4.6}$$

On the other hand, if $\text{ind}(v^0, 0)$ is an integer, then by Remark 3.2 the number n of interior orbits of v^0 connected to $\bar{x} = 0$ is even. Hence, one of the two boundary orbits of v^0 connected to $\bar{x} = 0$ is stable, and the other one is unstable. It follows that the exponent k in (4.6) is even, i.e., $k = 2m$ $(m \geq 1)$.

Consider the two equations

$$\begin{aligned} 0 &= v_1(x_1, 0, t_0 + \varepsilon) = v_1^0(x_1, 0) + \varepsilon v_1^1(x_1, 0) + o(|\varepsilon|) \\ &= \alpha x_1^{2m} + \varepsilon\beta + \varepsilon g(x_1) + o(|\varepsilon|, |x_1|^{2m}), \end{aligned} \tag{4.7}$$

$$\begin{aligned} 0 &= v_1(x_1, 0, t_0 - \varepsilon) = v_1^0(x_1, 0) - \varepsilon v_1^1(x_1, 0) + o(|\varepsilon|) \\ &= \alpha x_1^{2m} - \varepsilon\beta - \varepsilon g(x_1) + o(|\varepsilon|, |x_1|^{2m}), \end{aligned} \tag{4.8}$$

where $m \geq 1$. Without loss of generality, assume that $\alpha, \beta > 0$. Then there is a $\delta_0 > 0$ such that for any $\varepsilon > 0$ sufficiently small, (4.8) has only two solutions $x_1^{\pm}$ of opposite sign, $x_1^-(\varepsilon) < 0 < x_1^+(\varepsilon)$, in the interval $(-\delta_0, \delta_0)$, and (4.7) has no solutions in $(-\delta_0, \delta_0)$. The claim is verified. □

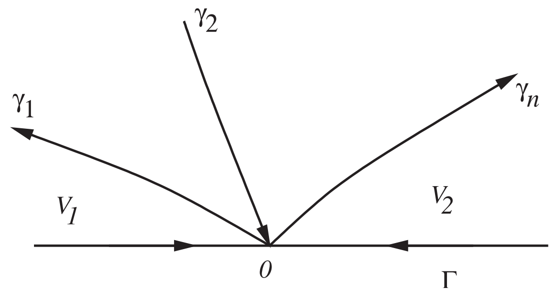

FIG. 4.1. *Case of n interior orbits connected to $\bar{x} = 0 \in \Gamma$, where $\Gamma \subset \partial M$ is flat, and $n \geq 3$ is odd.*

PROPOSITION 4.3. *If* $\text{ind}(v^0, 0) =$ *noninteger, then for any* $\varepsilon > 0$ *sufficiently small each* $u(x, t_0 \pm \varepsilon)$ *has exactly one* ∂*-singular point on* Γ.

Proof. Indeed, when $\text{ind}(v^0, 0) = -n/2$ is a fraction, then n is odd. Hence the exponent k in (4.6) is odd, i.e., $k = 2m + 1$ ($m \geq 1$). Therefore each of the two equations

$$\alpha x_1^{2m+1} + \varepsilon\beta + \varepsilon g(x_1) + o\left(|\varepsilon|, |x_1|^{2m+1}\right) = 0,$$
$$\alpha x_1^{2m+1} - \varepsilon\beta - \varepsilon g(x_1) + o\left(|\varepsilon|, |x_1|^{2m+1}\right) = 0$$

has exactly one solution in $(-\delta_0, \delta_0)$. ☐

4.2. Proof of Theorem 3.6. As mentioned earlier, u and v have topologically equivalent streamlines in an interior neighborhood of $x = 0$; hence bifurcation in the local structure of u at $\bar{x} = 0$ is equivalent to that of v. As a result, we only have to consider the local bifurcation for the vector field v. We divide the proof into two steps.

Step 1. *The case where* $\text{ind}(v^0, 0) =$ *integer.* By Theorem 3.5, the number of boundary saddle points of $v^0 + \varepsilon v^1$ in a small neighborhood $\Gamma_0 \subset \Gamma \subset \partial M$ of $\bar{x}$ is not the same as that for $v^0 - \varepsilon v^1$. Therefore, $v^0 + \varepsilon v^1$ and $v^0 - \varepsilon v^1$ are not topologically equivalent locally near $\bar{x} \in \Gamma \subset \partial M$.

Step 2. *The case where* $\text{ind}(v^0, 0) =$ *fraction.* Without loss of generality, we assume that the two orbits connected to $\bar{x} = 0$ on $\Gamma_0 \subset \partial M$ are stable (i.e., $\alpha < 0$ in (4.6)), and $v_1^1(0) > 0$ (i.e., $\beta > 0$ in (4.6)). Let $x^+ = (x_1^+, 0)$ and $x^- = (x_1^-, 0) \in \Gamma_0 \subset \partial M$ be the singular points of $v^0 + \varepsilon v^1$ and $v^0 - \varepsilon v^1$, respectively. Hence, $x_1^- < 0 < x_1^+$ as in Theorem 3.5. The nondegeneracy of both x^- and x^+ implies that

$$\text{ind}(v^0 \pm \varepsilon v^1, x^\pm) = -\frac{1}{2},$$

and there is exactly one orbit $\gamma^+(\varepsilon)$ of $v^0 + \varepsilon v^1$ in $\overset{\circ}{M}$ connected to x^+ (respectively, exactly only one orbit $\gamma^-(\varepsilon)$ of $v^0 - \varepsilon v^1$ in $\overset{\circ}{M}$ connected to x^-).

By (3.5) and Remark 3.2, there are n ($n \geq 3$ and odd) orbits γ_i ($1 \leq i \leq n$) of v^0 in $\overset{\circ}{M}$ connected to $\bar{x} \in \Gamma$; see Figure 4.1. Let $V_1 \subset \overset{\circ}{M}$ (respectively, $V_2 \subset \overset{\circ}{M}$) be the domain near $\bar{x} = 0 \in \Gamma_0 \subset \partial M$ enclosed by ∂M and γ_1 (respectively, by ∂M and γ_n).

Since $v_1^1(0) > 0$, the flow of $v^0 + \varepsilon v^1$ crosses γ_n transversally and enters into V_2 (respectively, the flow of $u^0 - \varepsilon u^1$ crosses γ_1 transversally and enters into V_1).

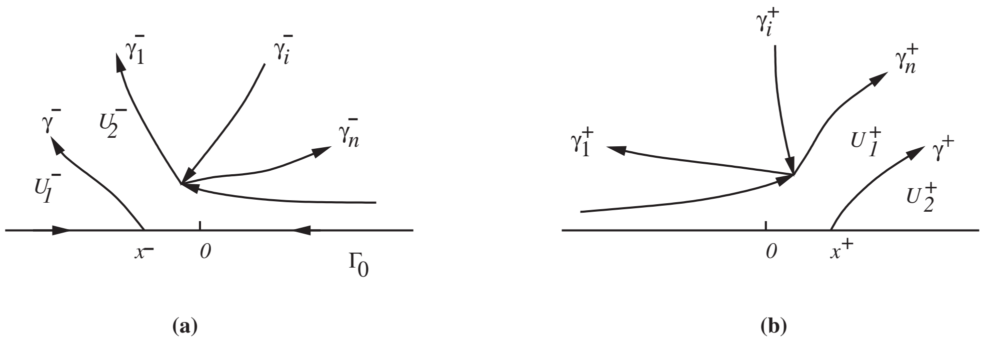

FIG. 4.2. *Sketch illustrating the proof of Assertion* (1) *in Theorem* 3.6*:* (a) *orbits at* $t^- < t_0$*, and* (b) *orbits at* $t^+ > t_0$

Therefore, we have

$$\gamma^+(\varepsilon) \subset V_2, \quad \gamma^-(\varepsilon) \subset V_1. \tag{4.9}$$

Obviously,

$$\lim_{\varepsilon\to 0} \gamma^-(\varepsilon) = \gamma_1, \quad \lim_{\varepsilon\to 0} \gamma^+(\varepsilon) = \gamma_n.$$

Moreover, there are n orbits $\gamma_i^+ = \gamma_i^+(\varepsilon)$ of $v^0 + \varepsilon v^1$ in $V - V_1$ (respectively, n orbits $\gamma_i^- = \gamma_i^-(\varepsilon)$ of $v^0 - \varepsilon v^1$ in $V - V_2$), which are connected to singular points of $v^0 + \varepsilon v^1$ (respectively, $v^0 - \varepsilon v^1$) such that

$$\lim_{\varepsilon\to 0} \gamma_i^+(\varepsilon) = \gamma_i, \qquad \lim_{\varepsilon\to 0} \gamma_i^-(\varepsilon) = \gamma_i,$$

where $V \subset \overset{\circ}{M}$ is a neighborhood of $\bar{x} = 0$.

Let U_1^- (respectively, U_2^+) be the domain enclosed by ∂M and γ^- (respectively, by ∂M and γ^+), and $U_2^- = V - U_1^-$ (respectively, $U_1^+ = V - U_2^+$); see Figures 4.2(a) and 4.2(b), respectively. We infer from (4.9) that

$$\gamma_i^+ \subset U_1^+, \quad \gamma_i^- \subset U_2^- \quad (1 \le i \le n). \tag{4.10}$$

We know that, under topological equivalence, the orbits of $v^0 + \varepsilon v^1$ connected to a singular boundary point are mapped, preserving orientation, to the orbits of $v^0 - \varepsilon v^1$ connected to a singular boundary point. Since $v^0 + \varepsilon v^1$ and $v^0 - \varepsilon v^1$ are topologically equivalent locally at $\bar{x} \in \Gamma$, the restriction of $v^0 + \varepsilon v^1$ to U_1^+ would have to be topologically equivalent to $v^0 - \varepsilon v^1$ in U_1^-. This is in contradiction with (4.10). Thus, assertion (1) of Theorem 3.6 is proven.

Assertion (2) of Theorem 3.6 is a corollary of assertion (1). Indeed, if $\bar{x} \in \partial M$ is a unique singular point of u^0 which has the same index as $\text{ind}(v^0, 0)$ on ∂M, then the structural bifurcation of $u(x,t)$ locally at $(\bar{x}, t_0)$ implies the structural bifurcation in its global structure.

The proof of Theorem 3.6 is thus complete.

5. Structural bifurcations near a curved boundary.

5.1. Main theorems. We now generalize in this section the main bifurcation theorems in section 3 for the flat boundary case to the curved boundary case.

Consider structural bifurcation near a ∂-singular point $\bar{x} \in \partial M$ of $u(x,t)$ on a general C^{r+1} boundary ∂M $(r \geq 2)$. Let $\bar{x} \in \partial M$ and (x_1, x_2) be an orthogonal coordinate system with origin $\bar{x}$, which has its x_1-axis tangent to ∂M at $\bar{x}$, and its x_2-axis oriented in the inward normal direction.

Let $u \in C^1([0,T], B_0^r(TM))$ $(r \geq 2)$ have the Taylor expansion at t_0 $(0 < t_0 < T)$ as in (3.2). In particular, let

$$u^0 = (u_1^0, u_2^0) = u(x, t_0),$$
$$u^1 = (u_1^1, u_1^2) = \frac{\partial u(x,t_0)}{\partial t}.$$

In addition, let n be the number of interior orbits of $u^0(x)$ connected to $\bar{x}$. Then we can restate Assumption (H) as (H′) below, with condition (3.5) on the index there replaced by a geometrical condition $n \neq 1$, i.e., (5.2) below.

Assumption (H′). Let $\bar{x} \in \partial M$ be an isolated degenerate ∂-singular point of $u^0(x) = u(x,t_0)$, with $u^0 \in C^{k+1}$ near $\bar{x} \in \Gamma$ for some $k \geq 2$. Assume that

$$\frac{\partial u^0(\bar{x})}{\partial n} = 0, \tag{5.1}$$
$$n \neq 1, \tag{5.2}$$
$$\frac{\partial u_1^1(\bar{x})}{\partial n} \neq 0, \tag{5.3}$$
$$\frac{\partial^{k+1} u_1^0(\bar{x})}{\partial^k \tau \partial n} \neq 0. \tag{5.4}$$

We have then the following structural bifurcation theorems, as in section 3.

THEOREM 5.1. *Let $u \in C^1([0,T], B_0^r(TM))$ satisfy Assumption* (H′) *and let $r \geq 2$. Then in a neighborhood $\Gamma \subset \partial M$ of $\bar{x}$, the ∂-singular points of $u(x, t_0 \pm \varepsilon)$ are nondegenerate for any $\varepsilon > 0$ sufficiently small. Moreover,*

1. *if $n =$ even $(n \geq 0)$, then one of $u(x, t_0 \pm \varepsilon)$ has two ∂-singular points on Γ, and the other one has no ∂-singular point on Γ; and*
2. *if $n =$ odd $(n \geq 3)$, then each of $u(x, t_0 \pm \varepsilon)$ has only one ∂-singular point on Γ.*

THEOREM 5.2 (structural bifurcation theorem). *Let $u \in C^1([0,T], B_0^r(TM))$ be a one-parameter family of divergence-free vector fields satisfying Assumption* (H′) *and $r \geq 2$. Then the following assertions hold true:*

1. *$u(x,t)$ has a bifurcation in its local structure at $(\bar{x}, t_0)$; and*
2. *if $\bar{x} \in \partial M$ is a unique degenerate ∂-singular point of $u^0(x) = u(x,t_0)$ on ∂M, then $u(x,t)$ has a bifurcation in its global structure at t_0.*

5.2. Coordinate transformation. The main ideas to prove Theorems 5.1 and 5.2 are as follows. First, we introduce a local coordinate transformation, which preserves the divergence-free character of the vector field and maps a neighborhood $\Gamma \subset \partial M$ of $\bar{x}$ to a flat boundary. This allows us to show that Assumption (H′) is equivalent to Assumption (H) for the new transformed vector field. Then Theorems 5.1 and 5.2 follow immediately from Theorems 3.5 and 3.6.

In the coordinate system (x_1, x_2) introduced at the beginning of this section, the boundary ∂M can be expressed locally near $\bar{x} = 0 \in \partial M$ by

$$x_2 = f(x_1), \quad f(0) = 0, \quad f'(0) = 0. \tag{5.5}$$

We make the local coordinate transformation

$$\begin{cases} \tilde{x}_1 = x_1, \\ \tilde{x}_2 = x_2 - f(x_1). \end{cases} \tag{5.6}$$

Obviously, the transformation (5.6) takes a neighborhood $U \subset M$ of $\bar{x}$ to a domain $\tilde{U} \subset \mathbb{R}^2_+ = \{(\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2 \mid \tilde{x}_2 \geq 0\}$ and maps the boundary part $U \cap \partial M$ to a neighborhood of $x = 0$ on the $\tilde{x}_1$-axis.

Let $\varphi : U \to \tilde{U}$ be the transformation (5.6), and $\varphi^* : C^r(TU) \to C^r(T\tilde{U})$ the isomorphism induced by φ. It is easy to see that

$$\varphi^* = D\varphi = \begin{pmatrix} 1 & 0 \\ -f'(\tilde{x}_1) & 1 \end{pmatrix},$$

and, for any $u \in C^r(TU)$,

$$\tilde{u} = \varphi^* \circ u = \begin{pmatrix} u_1 \\ u_2 - f'(\tilde{x}_1)u_1 \end{pmatrix}. \tag{5.7}$$

Then it is a direct calculation to derive the following lemma.

Lemma 5.3. *If $u \in C^r(TU)$ is divergence-free, then $\tilde{u} = \varphi^* \circ u$ is also divergence-free. Moreover, as $u|_{\partial M \cap U} = 0$, then $\tilde{u}(\tilde{x}_1, 0) = 0$, and*

$$\begin{cases} \tilde{u}(\tilde{x}) = \tilde{x}_2 \tilde{v}(\tilde{x}), \\ \tilde{v}_2(\tilde{x}_1, 0) = 0. \end{cases} \tag{5.8}$$

5.3. Proof of Theorems 5.1 and 5.2. According to Theorems 3.5 and 3.6, the proof of these two theorems will be achieved in a few lemmas as follows.

Lemma 5.4. *Let $u \in C^1([0,T], B^r_0(TM))$ satisfy Assumption* (H′). *Then the vector field $\tilde{u} = \varphi^* \circ u = \tilde{x}_2 \tilde{v}(\tilde{x}, t)$ satisfies Assumption* (H).

Proof. Notice that φ maps $\bar{x} = 0$ to $\tilde{x} = 0$. Since u and $\tilde{u}$ are topologically equivalent locally near $x = 0$ and $\tilde{x} = 0$, (5.2) implies that the number n of interior orbits connected to $\tilde{x} = 0$ of $\tilde{v}(x, t_0)$ is different from 1. Hence

$$\text{ind}(\tilde{v}(x, t_0), 0) = -\frac{n}{2} \neq -\frac{1}{2}.$$

By (5.7), $\tilde{u}_1 = u_1$, and φ^* takes the inward normal vector n at $x = 0$ to the normal vector $\tilde{n} = (0, 1)$ at $\tilde{x} = 0$. Therefore, we have

$$\frac{\partial u_1^1(0)}{\partial n} \neq 0 \Longrightarrow \frac{\partial \tilde{u}_1^1(0)}{\partial x_2} \neq 0,$$

where $u_1^1(x) = \partial u(x, t_0)/\partial t$.

By tensor analysis, we know that, under a coordinate transformation $\varphi : U \to \tilde{U}$, the directional derivative of a function $f(x)$ satisfies $\partial f/\partial r = \partial f/\partial \tilde{r}$, where r is a vector and $\tilde{r} = (D\varphi)r$.

By (5.7) we see that $u_1^0 = \tilde{u}_1^0$, and $\tilde{n} = (0, 1)$ at $\tilde{x} = 0$. Hence we have

$$\frac{\partial^k}{\partial \tau^k} \frac{\partial u_1^0(0)}{\partial n} = \frac{\partial^k}{\partial \tilde{\tau}^k} \cdot \frac{\partial \tilde{u}_1^0(0)}{\partial \tilde{n}} = \frac{\partial^k}{\partial x_1^k} \frac{\partial \tilde{u}_1^0(0)}{\partial x_2},$$

and the proof of the lemma is complete. □

LEMMA 5.5. *A point $x \in \partial M \cap U$ is a ∂-regular point of $u \in B_0^r(TM)$ if and only if $\tilde{x} = \varphi(x) = (\tilde{x}_1, 0)$ is a ∂-regular point of $\tilde{u} = D\varphi \cdot u$.*

Proof. By Definition 2.5, it suffices to show that the two vector fields $\partial u/\partial n$ and $\partial \tilde{u}/\partial \tilde{x}_2$ have the same singular points on the boundary in the sense of homeomorphism; here

$$\frac{\partial u}{\partial n} = (N \cdot \nabla)u = n_1 \frac{\partial u}{\partial x_1} + n_2 \frac{\partial n}{\partial x_2}$$

and the vector field N is defined in U with the unit modulus $|N| = 1$ such that the orbits of N are the normal lines λn in U. Note that, when U is properly chosen, for any $x, y \in U \cap \partial M$, $x \neq y$, the normal lines λn_x and λn_y do not intersect within U.

Equivalently, we proceed to check the desired result for the two vector fields $\partial \tilde{u}/\partial \tilde{n}$ and $\partial \tilde{u}/\partial \tilde{x}_2$, where $\partial \tilde{u}/\partial \tilde{n}$ is the transformation of $\partial u/\partial n$ that is expressed by

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial \tilde{n}} = \left(\tilde{N} \cdot \tilde{\nabla}\right) \tilde{u} = \tilde{n}_1 \dfrac{\partial \tilde{u}}{\partial \tilde{x}_1} + \tilde{n}_2 \dfrac{\partial \tilde{u}}{\partial \tilde{x}_2}, \\ \tilde{N} = \begin{pmatrix} \tilde{n}_1 \\ \tilde{n}_2 \end{pmatrix} = D\varphi \circ N = \begin{pmatrix} 1 & 0 \\ -f' & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 - f' n_1 \end{pmatrix}. \end{cases} \tag{5.9}$$

From (5.8) we see that

$$\tilde{u}(\tilde{x}_1, 0) = 0 \quad \forall\ \tilde{x}_1 \in \partial \mathbb{R}_+^2 \cap \tilde{U}. \tag{5.10}$$

By (5.9) and (5.10) we deduce that

$$\frac{\partial \tilde{u}}{\partial \tilde{n}} = (n_2 - f'(x_1)n_1) \frac{\partial \tilde{u}}{\partial \tilde{x}_2} \quad \text{ on } \partial \mathbb{R}_+^2 \cap \tilde{U}. \tag{5.11}$$

From (5.5) and $n_2 \neq 0$ for $x \in \partial M$ near $\bar{x}$, the result follows and the proof of the lemma is complete. □

LEMMA 5.6. *A point $x \in \partial M \cap U$ is a ∂-saddle point of $u \in B_0^r(TM)$ if and only if $\tilde{x} = \varphi(x) \in \partial \mathbb{R}_+^2 \cap \tilde{U}$ is a ∂-saddle point of $\tilde{u} = D\varphi \cdot u$.*

Proof. It suffices to prove that $\partial \tilde{u}/\partial \tilde{n}$ and $\partial \tilde{u}/\partial \tilde{x}_2$ have the same nondegenerate singular points on the boundary $\Gamma = \partial \mathbb{R}_+^2 \cap \tilde{U} = \{(\tilde{x}_1, 0) \mid |\tilde{x}_1| < \delta\}$, i.e., both Jacobian determinants

$$\det \begin{pmatrix} \dfrac{\partial^2 \tilde{u}_1}{\partial \tilde{x}_1 \partial \tilde{n}} & \dfrac{\partial^2 \tilde{u}_1}{\partial \tilde{x}_2 \partial \tilde{n}} \\ \\ \dfrac{\partial^2 \tilde{u}_2}{\partial \tilde{x}_1 \partial \tilde{n}} & \dfrac{\partial^2 \tilde{u}_2}{\partial \tilde{x}_2 \partial \tilde{n}} \end{pmatrix} \quad \text{and} \quad \det \begin{pmatrix} \dfrac{\partial^2 \tilde{u}_1}{\partial \tilde{x}_1 \partial \tilde{x}_2} & \dfrac{\partial^2 \tilde{u}_1}{\partial \tilde{x}_2^2} \\ \\ \dfrac{\partial^2 \tilde{u}_2}{\partial \tilde{x}_1 \partial \tilde{x}_2} & \dfrac{\partial^2 \tilde{u}_2}{\partial \tilde{x}_2^2} \end{pmatrix}$$

have the same nonzero points on Γ.

From (5.8) we see that

$$\frac{\partial \tilde{u}_2(\tilde{x}_1, 0)}{\partial x_2} = 0 \quad \forall\, |x_1| < \delta \tag{5.12}$$

for some $\delta > 0$. By (5.11) and (5.12) one deduces that

$$\frac{\partial^2 \tilde{u}_2(\tilde{x}_1, 0)}{\partial \tilde{x}_1 \partial \tilde{n}} = 0, \qquad \frac{\partial^2 \tilde{u}_2(\tilde{x}_1, 0)}{\partial \tilde{x}_1 \partial \tilde{x}_2} = 0.$$

Hence we only need to prove that

$$\frac{\partial^2 \tilde{u}_1(\tilde{x}_1, 0)}{\partial \tilde{x}_1 \partial \tilde{n}} \neq 0 \Longleftrightarrow \frac{\partial^2 \tilde{u}_1(\tilde{x}_1, 0)}{\partial \tilde{x}_1 \partial \tilde{x}_2} \neq 0, \tag{5.13}$$

$$\frac{\partial^2 \tilde{u}_2(\tilde{x}_1, 0)}{\partial \tilde{x}_2 \partial \tilde{n}} \neq 0 \Longleftrightarrow \frac{\partial^2 \tilde{u}_2(\tilde{x}_1, 0)}{\partial \tilde{x}_2^2} \neq 0. \tag{5.14}$$

From (5.11) and (5.12) we immediately derive (5.14).

Thanks to (5.8), for any integer $k \geq 0$,

$$\frac{\partial^k \tilde{u}_1(\tilde{x}_1, 0)}{\partial \tilde{x}_1 k} = 0. \tag{5.15}$$

By assumption, $(\tilde{x}_1, 0)$ is a singular point of $\partial \tilde{u}/\partial x_2$, i.e., a ∂-saddle point of $\tilde{u}$:

$$\frac{\partial \tilde{u}_1(\tilde{x}_1, 0)}{\partial x_2} = 0. \tag{5.16}$$

From (5.15) and (5.16) it follows that

$$\begin{aligned} \frac{\partial^2 \tilde{u}_1(\tilde{x}_1, 0)}{\partial \tilde{x}_1 \partial \tilde{n}} &= \frac{\partial}{\partial \tilde{x}_1} \left[\tilde{n}_1 \frac{\partial \tilde{u}_1}{\partial \tilde{x}_1} + \tilde{n}_2 \frac{\partial \tilde{u}_1}{\partial \tilde{x}_2} \right] \Bigg|_{\tilde{x} = (\tilde{x}_1, 0)} \\ &= \tilde{n}_2 \, \frac{\partial \tilde{u}_1(\tilde{x}_1, 0)}{\partial \tilde{x}_1 \partial \tilde{x}_2} \ . \end{aligned}$$

By (5.9), $\tilde{n}_2 = n_2 - f'(x_1) n_1 \neq 0$ for $x = (x_1, x_2) \in \partial M$ near $\bar{x} = 0$. Thus we derive (5.13), and the proof is complete. □

6. Applications to boundary-layer separation.

6.1. Two examples. In order to understand intuitively the connection between structural bifurcation and boundary-layer separation, we proceed by discussing two typical examples that illustrate how structural bifurcations occur in some fluid flows. For simplicity, we consider in this section only bifurcation near flat boundaries.

Let $u \in C^1([0, T], B_0^r(TM))$, $\bar{x} \in \Gamma \subset \partial M$ be an isolated ∂-singular point of $u^0(x) = u(x, t_0)$, $0 < t_0 < T$, where Γ is a flat part of ∂M. We take a coordinate system (x_1, x_2) with $\bar{x}$ at the origin and $\Gamma = \{x_1, 0) \,\big|\, |x_1| < \delta\}$. Thus $u(x, t)$ can be expressed in a neighborhood $U \subset M$ of $\bar{x}$ by

$$u(x, t) = x_2 v(x, t), \tag{6.1}$$

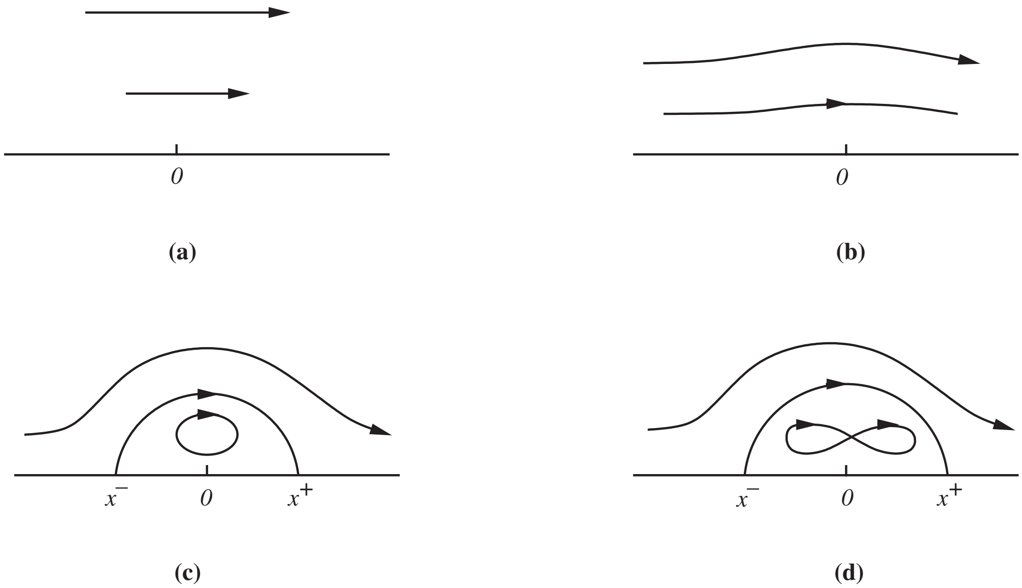

FIG. 6.1. *Boundary-layer separation and reattachment near a flat boundary.*

as in section 3. Let

$$v^0(x) = v(x, t_0) = (v_1^0(x), v_2^0(x)),$$
$$v^1(x) = \frac{\partial}{\partial t} v(x, t_0) = (v_1^1(x), v_2^1(x)).$$

Example 6.1. When the index of the vector field $v^0(x)$ at the singular point $\bar{x} = 0$ is zero, i.e., $\text{ind}(v^0, 0) = 0$, structural bifurcation occurs, as shown in Figure 6.1. The figure corresponds to boundary-layer separation of an incompressible, initially parallel flow over a flat plate.

Figure 6.1(a) shows the flows structure of $u(x, t_0 - \varepsilon)$, a typical shear flow near the boundary. At the time instant $t_0 - \varepsilon$, with $\varepsilon > 0$ small, the flow exhibits no singular points near $\bar{x} = 0$. At the time instant t_0, the time at which structural bifurcation occurs, $u^0(x) = u(x, t_0)$ is given by Figure 6.1(b), which has an isolated ∂-singular point $\bar{x} = 0 \in \partial M$. At a later time, $u(x, t_0 + \varepsilon)$ is given by either Figure 6.1(c) or Figure 6.1(d). Even more complicated flow patterns are possible in the recirculation region, but in a real fluid the figure-eight streamline in Figure 6.1(d) will be affected by the viscosity and evolve into two separate, counterrotating vortices. On the boundary, there are exactly two ∂-saddle points on ∂M near $\bar{x} = 0$, denoted by x^- and x^+ in Figures 6.1(c) and 6.1(d). We shall prove hereafter that the pattern shown in Figure 6.1(c) is generic.

Example 6.2. When $\text{ind}(v^0, 0) = -1$, global structural bifurcation of u may occur, due to a local transition near $\bar{x} \in \partial M$. An example of such a global bifurcation of an incompressible flow field is shown in Figure 6.2. The transition from Figure 6.2(a) through Figure 6.2(b) to Figure 6.2(c) is more idealized than in Figure 6.1 but is entirely consistent with our rigorous results, as well as physically plausible.

6.2. Boundary-layer separation. We now address the separation of streamlines and their reattachment in a 2-D divergence-free vector field, from the point of

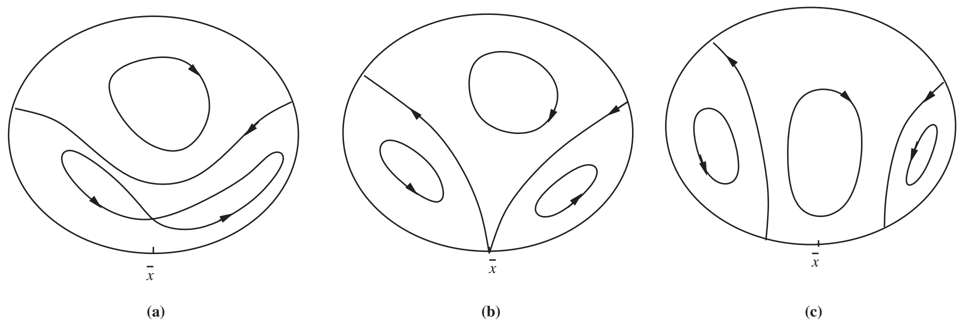

FIG. 6.2. *Global structural bifurcation of a* 2*-D incompressible flow in a compact domain.*

view of the rigorous results in sections 3 and 5. The connection to actual boundary-layer separation in an incompressible fluid governed by the Navier–Stokes equations is made in the companion papers [7, 12].

We start with Assumption (H) in the case where ind $(v^0, 0) = 0$, i.e., there are no interior orbits of u^0 connected to $\bar{x} = 0$. For convenience, we call it Assumption (H_0), and it reads as follows.

Assumption (H_0). Let $\bar{x} = 0 \in \Gamma$ be an isolated degenerate ∂-singular point of $u^0(x)$, $u^0 \in C^{k+1}$ near $\bar{x} \in \Gamma$ for some $k \geq 2$. Assume that

$$\frac{\partial u^0(0)}{\partial n} = 0, \tag{6.2}$$

$$\text{ind}(v^0, 0) = 0, \tag{6.3}$$

$$\frac{\partial u^1(0)}{\partial n} \neq 0, \tag{6.4}$$

$$\frac{\partial^{k+1} u^0_1(0)}{\partial^k \tau \partial n} \neq 0. \tag{6.5}$$

THEOREM 6.3. *Let $u \in C^1([0,T], B^r_0(TM))$ be as given in* (6.1) *satisfying Assumption* (H_0). *Then the following conclusions hold:*

1. *There must be some closed orbits of u separated from $\bar{x} = 0 \in \partial M$, as shown schematically in either Figure* 6.1(c) *or Figure* 6.1(d).
2. *The flow structure after the separation enjoys the following properties:*
   (a) *there are exactly two ∂-saddle points x^- and x^+ of $u(\cdot, t)$ near $\bar{x} = 0$ with one on each side of $\bar{x}$, $x^- < \bar{x} < x^+$;*
   (b) *x^- and x^+ are connected by an extended interior orbit $\gamma(t)$ that consists of orbits of $u(\cdot, t)$; and*
   (c) *the closed orbits of conclusion* (1) *above are enclosed by the extended orbit $\gamma(t)$ and the portion of the boundary between x^- and x^+.*
3. *The whole extended orbit $\gamma(t)$ shrinks to $\bar{x}$ as $t \to t_0$.*

A few remarks are now in order.

Remark 6.4. The closed orbits in Figures 6.1(c), 6.1(d), and 6.2(a)–(c) correspond, in a real fluid, to isolated vortices or, in the case of figure-eight ones, to pairs of counterrotating vortices.

Remark 6.5. The separation occurs as t crosses the critical instant t_0 from either left to right or from right to left; this is dictated by the orientation of u^1 in comparison to that of u^0, in the expansion (3.2).

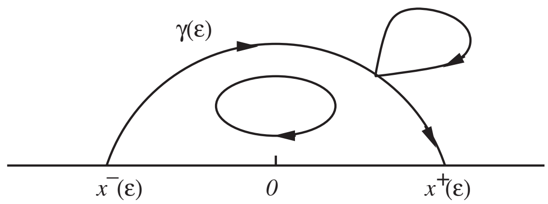

FIG. 6.3. *Convergence of an extended orbit* $\gamma(\varepsilon) \to \bar{x} = 0$.

Remark 6.6. The reattachment of a streamline $\gamma(t)$ to the boundary, as described in conclusion 2(b) above, differs from a particle's streakline (i.e., from its path in real time): as the vortex grows in time, particles are either trapped inside the vortex or pushed into the interior of the fluid, away from the vortex.

Proof of Theorem 6.3. By Theorem 3.5, without loss of generality, we assume that $u(x, t_0+\varepsilon)$ has two ∂-saddle points $x^-(\varepsilon)$ and $x^+(\varepsilon)$, and $u(x, t_0-\varepsilon)$ has none. Then both ∂-saddle points $x^-(\varepsilon)$ and $x^+(\varepsilon)$ of $u(x, t_0+\varepsilon)$ tend to $x = 0$ as $\varepsilon \to 0$.

By assumption, the singular point $\bar{x} \in \partial M$ of $v^0(x)$ is isolated; therefore the stability lemma of extended orbits (Lemma 7.2) in the appendix ensures that both ∂-saddle points $x^-(\varepsilon)$ and $x^+(\varepsilon)$ must be connected by an extended orbit $\gamma(\varepsilon)$ with $\gamma(\varepsilon) \to \bar{x}$ as $\varepsilon \to 0$; see Figure 6.3. Because the sum of the indices of the singular points near $\bar{x} \in \partial M$ of $u(x, t_0 \pm \varepsilon)$ is zero, there exist centers of $v(x, t_0+\varepsilon)$ near $\gamma(\varepsilon)$, which converge to $\bar{x} \in \partial M$ as $\varepsilon \to 0$. The other assertions are even easier to verify.

In the above theorem, there might be several centers that appear in the recirculation region. However, subject to an additional but generic assumption, the following theorem shows that there must be exactly one center that separates from the boundary near $\bar{x}$.

THEOREM 6.7. *Let* $u \in C^1([0,T], B_0^r(TM))$ *be as given in* (6.1)*; satisfy Assumption* (H$_0$) *with* $k = 2$, *and*

$$\frac{\partial^2 u_1^0(0)}{\partial x_2^2} \neq 0. \tag{6.6}$$

Then the center separated from $\bar{x} \in \partial M$ *is unique, as shown in Figure* 6.1(c).

Remark 6.8. Consider the Taylor expansion of the vorticity $\omega = -\partial u_2/\partial x_1 + \partial u_1/\partial x_2$ of the vector field u:

$$\begin{cases} \omega(x,t) = \omega^0(x) + \omega^1(x)(t-t_0) + o((t-t_0)^2), \\ \omega^0(x) = \omega(x,t_0), \qquad \omega^1(x) = \dfrac{\partial \omega}{\partial t}(x,t_0). \end{cases} \tag{6.7}$$

Conditions (6.2)–(6.4), and (6.6) are equivalent, respectively, to

$$\omega^0(0) = 0, \tag{6.8}$$

$$\frac{\partial^2 \omega^0(0)}{\partial x_1^2} \neq 0, \tag{6.9}$$

$$\omega^1(0) \neq 0, \tag{6.10}$$

$$\frac{\partial \omega^0(0)}{\partial n} \neq 0. \tag{6.11}$$

Proof of Theorem 6.7. First we observe that conditions (6.2), (6.4), and (6.5) are equivalent to the following conditions on v:

$$\frac{\partial^2 v_1^0(0)}{\partial x_1^2} \neq 0, \tag{6.12}$$

$$\frac{\partial v_1^0(0)}{\partial x_2} \neq 0, \tag{6.13}$$

$$v_1^1(0) \neq 0. \tag{6.14}$$

We only have to prove that the interior singular point of $v(x, t_0+\varepsilon)$ near $\bar{x} \in \partial M$ is unique. By the Taylor expansion, we have

$$\begin{cases} v_1(x, t_0+\varepsilon) = v_1^0(x) + \varepsilon v_1^1(x) + o(|\varepsilon|), \\ v_2(x, t_0, \varepsilon) = v_2^0(x) + \varepsilon v_2^1(x) + o(|\varepsilon|). \end{cases}$$

By the nondivergence of $u^0(x)$ and by (6.12) and (6.13), we derive

$$\begin{cases} v_1^0(x) = \lambda x_2 + \alpha x_1^2 + o(|x_1|^2, |x_2|), \quad \alpha \neq 0,\, \lambda \neq 0, \\ v_2^0(x) = -\alpha x_1 x_2 + x_2 \cdot o(|x|). \end{cases}$$

By (6.14) we have

$$\begin{cases} v_1^1(x) = \beta + O(|x|), \quad \beta < 0, \\ v_2^1(x) = x_2 \cdot O(|x|). \end{cases}$$

Hence the interior singular points $(\tilde{x}_1, \tilde{x}_2)$ of $v(x, t_0+\varepsilon)$ with $\tilde{x}_2 > 0$ satisfy the equations

$$\begin{cases} \alpha x_1^2 + \lambda x_2 + \varepsilon\beta + o(|\varepsilon|, |x_2|, |x_1|^2) = 0, \\ -\alpha x_1 + \varepsilon \cdot O(|x|) + o(|x|) = 0, \\ \alpha \neq 0,\, \lambda \neq 0,\, \beta < 0. \end{cases} \tag{6.15}$$

It follows from the implicit function theorem that a solution $(\tilde{x}_1, \tilde{x}_2)$ with $\tilde{x}_2 > 0$, if it exists, is unique for any $\varepsilon > 0$ sufficiently small. The existence of such a solution to (6.15) is derived by Theorem 6.3, and the proof is complete. ∎

Remark 6.9. The set of all vector fields u satisfying (6.12) and (6.13) is open and dense in the topological space

$$A = \left\{ u \in C^1([0,T], B_0^3(TM)) \,\middle|\, u \text{ satisfy Assumption } (\mathrm{H}_0) \right\}.$$

Hence, Theorem 6.3 shows that the separation from the boundary of a simple vortex is generic.

7. Appendix. Extended orbits and their stability. The purpose of this appendix is to recall a lemma on stability of extended orbits [11]. We start with a definition.

DEFINITION 7.1. *Let $v \in C^r(TM)$ be a vector field. A curve $\gamma \subset M$ is called an extended orbit of v if*

(i) *it is a union of curves*

$$\gamma = \bigcup_{i=1} \gamma_i;$$

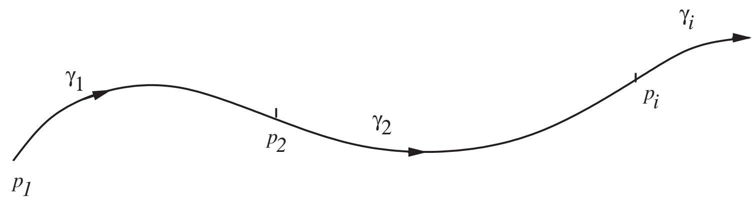

Fig. 7.1. *An extended orbit.*

(ii) *either γ_i is an orbit of v, or γ_i consists of both orbits and singular points of v; and*

(iii) *γ_i and γ_{i+1} are orbits of v; then the ω-limit set of γ_i is the α-limit set of γ_{i+1},*

$$\omega(\gamma_i) = \alpha(\gamma_{i+1}),$$

namely, the end points of γ_i are singular points of v, and the starting end point of γ_{i+1} is the finishing end point of γ_i; see Figure 7.1.

The point $p_1 = \alpha(\gamma_1)$ is called the starting point of the extended orbit γ.

The following stability lemma for extended orbits has been proved by Ma and Wang [11] in Step 2 of their proof of Lemma 4.5. We restate it here as a separate lemma since it is quite useful in analyzing the orbits of families of vector fields, and thus in solving some problems in 2-D incompressible fluid flows.

Lemma 7.2 (stability of extended orbits [11]). *Let $v^n \in C^r(TM)$ be a sequence of 2-D vector fields with $\lim_{n\to\infty} v^n = v \in C^r(TM)$. Suppose that $\gamma^n \subset M$ is an extended orbit of v^n and the starting points p_1^n of γ^n converge to p_1. Then the extended orbits γ^n of v^n converge to an extended orbit γ of v with starting point p_1.*

Acknowledgments. The authors are grateful to an anonymous referee for his/her insightful comments. Preliminary results of this investigation were presented at the the Fourth International Conference on Dynamical Systems and Differential Equations held in Wilmington, North Carolina, in May 2002, and at the SIAM Conference on Applications of Dynamical Systems (DS03) held in Snowbird, Utah, in May 2003.

REFERENCES

[1] L. Caffarelli, R. Kohn, and L. Nirenberg, *On the regularity of the solutions of Navier-Stokes equations*, Comm. Pure Appl. Math., 35 (1982), pp. 771–831.

[2] A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, Springer-Verlag, New York, 1997.

[3] P. Constantin and C. Foias, *The Navier-Stokes Equations*, University of Chicago Press, Chicago, 1988.

[4] W. E, *Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation*, Acta Math. Sin. (Engl. Ser.), 16 (2000), pp. 207–218.

[5] W. E and B. Engquist, *Blow-up of solutions of the unsteady Prandtl's equation*, Comm. Pure Appl. Math, 50 (1997), pp. 1287–1293.

[6] M. Ghil and S. Childress, *Topics in Geophysical Fluid Dynamics: Atmospheric Dynamics, Dynamo Theory, and Climate Dynamics*, Springer-Verlag, New York, 1987.

[7] M. Ghil, J.-G. Liu, C. Wang, and S. Wang, *Boundary-layer separation and adverse pressure gradient for 2-D viscous incompressible flow*, submitted.

[8] M. Ghil, T. Ma, and S. Wang, *Structural bifurcation of 2-D incompressible flows*, Indiana Univ. Math. J., 50 (2001), pp. 159–180.

[9] S. Goldstein, *Modern Developments in Fluid Dynamics, Vols.* I *and* II, Dover Publications, New York, 1965.

[10] T. Ma and S. Wang, *Structure of 2D incompressible flows with the Dirichlet boundary conditions*, Discrete Contin. Dyn. Syst. Ser. B, 1 (2001), pp. 29–41.

[11] T. Ma and S. Wang, *Structural classification and stability of incompressible vector fields*, Phys. D, 171 (2002), pp. 107–126.

[12] T. Ma and S. Wang, *Boundary layer separation and structural bifurcation for 2-D incompressible fluid flows*, Discrete Contin. Dyn. Syst. Ser. A, 10 (2003), pp. 459–472.

[13] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge, UK, 2002.

[14] O. Oleinik and V. N. Samokhin, *Mathematical Models in Boundary Layer Theory*, Chapman and Hall/CRC, Boca Raton, FL, 1999.

[15] L. Prandtl, in Verhandlungen des dritten internationalen Mathematiker-Kongresses, Heidelberg, 1904, Leipzig, 1905, pp. 484–491.

[16] H. Schlichting, *Boundary Layer Theory*, 8th ed., Springer-Verlag, Berlin, Heidelberg, 2000.

[17] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, 3rd ed., North-Holland, Amsterdam, 1984.

[18] X. Wang and R. Temam, *Asymptotic analysis of Oseen type equations in a channel at high Reynolds number*, Indiana Univ. Math. J., 45 (1996), pp. 863–916.